MIND – ATLAS

(collection of mind-maps to the Lecture on QFT)

Peskin-Schroeder
2. Klein-Gordon field

2.1 Why QFT

- *Particle creation not part of QM*
  Even below threshold, virtual states can be created due to Heisenberg uncertainty
- *Causality violation*
  When only particle exist, they can propagate btw. arbitrary points in arbitrary short time.

2.2 Classical FT

- Principle of least action

$$S = \int L\,dt = \int \mathcal{L}(\phi, \partial_\mu \phi)\, d^4x.$$

- *Euler-Lagrange* equation of motion

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0.$$

- *Hamiltonian*

$$H = \int d^3x \left[\pi(\mathbf{x}) \dot{\phi}(\mathbf{x}) - \mathcal{L} \right] \equiv \int d^3x\, \mathcal{H}.$$

- E.g.: interaction-less scalar theory

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 \Longrightarrow (\partial_\mu \partial^\mu + m^2)\phi = 0$$

- Symmetries/Conservation laws → *Noether's Theorem*
  a) take continuous transformation of the field

$$\phi(x) \to \phi'(x) = \phi(x) + \alpha \Delta\phi(x),$$

  b) it is a *symmetry*, iff

$$\mathcal{L}(x) \to \mathcal{L}(x) + \alpha \partial_\mu \mathcal{J}^\mu(x),$$

  so that the action does not change
  c) Conserved current:

$$\partial_\mu j^\mu(x) = 0, \qquad \text{for} \quad j^\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi - \mathcal{J}^\mu.$$

  E.g.: Space-time symmetry leads to conserved *Energy-momentum* tensor

$$T^\mu{}_\nu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \mathcal{L} \delta^\mu{}_\nu.$$

2.3 KG field as HO

- *Second quantization* = promote WF to operators.
  Like for x, p commutators, we generalize:

$$[\phi(\mathbf{x}), \pi(\mathbf{y})] = i\delta^{(3)}(\mathbf{x} - \mathbf{y});$$
$$[\phi(\mathbf{x}), \phi(\mathbf{y})] = [\pi(\mathbf{x}), \pi(\mathbf{y})] = 0.$$

- Klein-Gordon Hamiltonian

$$\phi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_\mathbf{p}}} \left(a_\mathbf{p} e^{i\mathbf{p}\cdot\mathbf{x}} + a_\mathbf{p}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}} \right);$$
$$\pi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_\mathbf{p}}{2}} \left(a_\mathbf{p} e^{i\mathbf{p}\cdot\mathbf{x}} - a_\mathbf{p}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}} \right).$$

- …

- Demanding Lorentz invariance, one particle state is

$$|\mathbf{p}\rangle = \sqrt{2E_\mathbf{p}}\, a_\mathbf{p}^\dagger |0\rangle,$$

Normalization:

$$\langle \mathbf{p} | \mathbf{q} \rangle = 2E_\mathbf{p} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}).$$

Completeness relation:

$$(1)_{1\text{-particle}} = \int \frac{d^3p}{(2\pi)^3} |\mathbf{p}\rangle \frac{1}{2E_\mathbf{p}} \langle \mathbf{p}|.$$

Lorentz invariant 3-integral:

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_\mathbf{p}} = \int \frac{d^4p}{(2\pi)^4} (2\pi)\delta(p^2 - m^2)\Big|_{p^0>0}$$

Scalar particle at point x is created as:

$$\phi(\mathbf{x})|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}} |\mathbf{p}\rangle$$

2.4 KG field in Space-Time

- positive-frequency solutions create a particle, negative ones destroy those

$$\phi(\mathbf{x}, t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_\mathbf{p}}} \left(a_\mathbf{p} e^{-ip\cdot x} + a_\mathbf{p}^\dagger e^{ip\cdot x} \right)\Big|_{p^0 = E_\mathbf{p}};$$
$$\pi(\mathbf{x}, t) = \frac{\partial}{\partial t} \phi(\mathbf{x}, t).$$

  consistent with QM requirement that only positive excitations exist
- *Propagator* from x to y

$$D(x - y) = \langle 0 | \phi(x) \phi(y) | 0 \rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_\mathbf{p}} e^{-ip\cdot(x-y)}.$$

- when *(x-y)^2<0*, causality preserved

$$[\phi(x), \phi(y)] = D(x - y) - D(y - x) = 0$$

  → existence of *antiparticles = same mass, inverse charge particles*
  → KG particles are real → they are their own antiparticles
- *Retarded propagator* (=takes only one time direction)

$$D_R(x - y) = \theta(x^0 - y^0) \langle 0 | [\phi(x), \phi(y)] | 0 \rangle.$$

  → *Green's function* of KG operator

$$(\partial^2 + m^2) D_R(x - y) = -i\delta^4(x - y)$$

  → *Feynman* prescription (guarantees the right time ordering)

$$D_F(x - y) = \int \frac{d^4}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)} = \langle 0 | T\phi(x)\phi(y) | 0 \rangle$$

- How new particles can be created?
  a) introduce a current *j(x)≠0 for* $t_1<t<t_2$

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \phi)^2 - \tfrac{1}{2} m^2 \phi^2 + j(x)\phi(x).$$

  b) total number of particles created

$$\int dN = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_\mathbf{p}} |\tilde{j}(p)|^2.$$

Chapter 3 – The Dirac Field

Lorentz Invariance – 3.1

We label a general Lorentz transformation by Λ. We define the scalar field to transform as

$$\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x).$$

We define a vector field to transform as

$$\Phi_a(x) \to M_{ab}(\Lambda)\Phi_b(\Lambda^{-1}x).$$

The Lorentz transformations Λ form a *group*. The $M(\Lambda)$ are the ***representations*** of the Lorentz group. The matrices forming the representation must satisfy

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\left(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}\right).$$

Dirac BiLinears – 3.4

	transformation property of $\bar\psi\Gamma\psi$	
1	scalar	1
γ^μ	vector	4
$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$	tensor	6
$\gamma^\mu\gamma^5$	pseudo-vector	4
γ^5	pseudo-scalar	1
		16

to write **any** 4x4 matrix in terms of gamma matrices.*(pseudo=parity-asym.)*

There are conserved currents

$$j^\mu(x) = \bar\psi(x)\gamma^\mu\psi(x)$$

$$j^{\mu 5}(x) = \bar\psi(x)\gamma^\mu\gamma^5\psi(x)$$

The Dirac Equation – 3.2

The Dirac algebra defined by

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \times \mathbf{1}_{n\times n}$$

allows us to define the following generators

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu].$$

We can then write the ***chiral/Weyl representation*** of Λ for 3+1 space-time using

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}.$$

The **spinor representation** of the Lorentz group is

$$\Lambda_{\frac{1}{2}} = \exp\left(-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)$$

The Dirac equation is

$$(i\gamma^\mu\partial_\mu - m)\psi(x) = 0.$$

We will generally use

$$\bar\psi \equiv \psi^\dagger\gamma^0$$

since then psibar-psi is a Lorentz scalar. The Dirac Lagrangian is

$$\mathcal{L}_{\text{Dirac}} = \bar\psi(i\gamma^\mu\partial_\mu - m)\psi.$$

We can use Weyl spinors to talk about left or right handed particles.

Free-Particle Solution – 3.3

Dirac equation also satisfies KG equation so use

$$\psi(x) = u(p)e^{-ip\cdot x}$$

$$u(p_0) = \sqrt{m}\begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

Apply a boost to get

$$u(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar\sigma}\,\xi \end{pmatrix}$$

With normalization condition

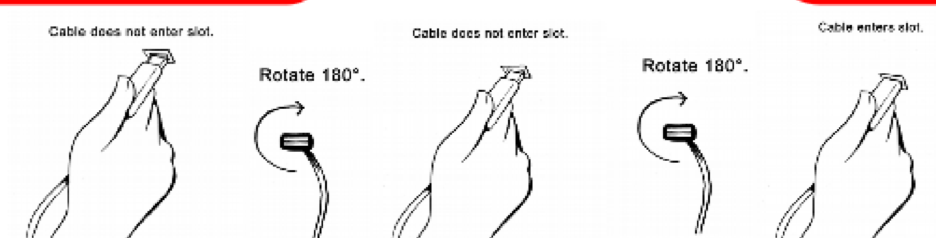
$$\bar u u = 2m\xi^\dagger\xi.$$

Spin sums (over ξ^s

$$\sum_s u^s(p)\bar u^s(p) = \gamma\cdot p + m.$$

$$\sum_s v^s(p)\bar v^s(p) = \gamma\cdot p - m.$$

PROVED:
Cables exist in 4-dimensional space.



Non-orientable MF?

Quantizing Dirac Field – 3.5

The quantized fields are

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_s\left(a^s_{\mathbf{p}}u^s(p)e^{-ip\cdot x} + b^{s\dagger}_{\mathbf{p}}v^s(p)e^{ip\cdot x}\right)$$

$$\bar\psi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_s\left(b^s_{\mathbf{p}}\bar v^s(p)e^{-ip\cdot x} + a^{s\dagger}_{\mathbf{p}}\bar u^s(p)e^{ip\cdot x}\right)$$

With commutation relations!!! **NOT [. , .]**

$$\{a^r_{\mathbf{p}}, a^{s\dagger}_{\mathbf{q}}\} = \{b^r_{\mathbf{p}}, b^{s\dagger}_{\mathbf{q}}\} = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\delta^{rs}$$

The one particle state is defined by

$$|\mathbf{p}, s\rangle \equiv \sqrt{2E_{\mathbf{p}}}\,a^{s\dagger}_{\mathbf{p}}|0\rangle$$

We can show the Dirac field is in fact a spin ½ particle by looking at the angular momentum

$$\mathbf{J} = \int d^3x\,\psi^\dagger\left(\mathbf{x}\times(-i\nabla) + \tfrac{1}{2}\mathbf{\Sigma}\right)\psi$$

Then on a state we get

$$J_z a^{s\dagger}_0|0\rangle = \pm\tfrac{1}{2}a^{s\dagger}_0|0\rangle, \qquad J_z b^{s\dagger}_0|0\rangle = \mp\tfrac{1}{2}b^{s\dagger}_0|0\rangle,$$

The total charge is

$$Q = \int \frac{d^3p}{(2\pi)^3}\sum_s\left(a^{s\dagger}_{\mathbf{p}}a^s_{\mathbf{p}} - b^{s\dagger}_{\mathbf{p}}b^s_{\mathbf{p}}\right).$$

The propagator is

$$\langle 0|\psi_a(x)\bar\psi_b(y)|0\rangle = (i\partial\!\!\!/_x + m)_{ab}\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-ip\cdot(x-y)}$$

$$\langle 0|\bar\psi_b(y)\psi_a(x)|0\rangle = -(i\partial\!\!\!/_x + m)_{ab}\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-ip\cdot(y-x)}$$

Section 4.1
Perturbation Theory

"phi-fourth"
$$L = \left(\frac{1}{2}\right)(\partial_\mu \phi))^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$$
QED Lagrangian

Gauge Invariance issues

Renormalizability considerations

Remember that L must have units of m^4
scalar and vector fields have units of m spinors have units of 3/2

Renormalizable terms must have Proper units, but dimensionless Coupling constants.

Section 4.2
Perturbation Expansion of Correlation Functions

Difference between the ground State and the vacuum

Interacting Correlator in the Hisenberg picture.

The time-evolution operator

The true description of the Vacuum

The perturbed Correlator in the Interaction picture.

Section 4.3
Wick's Theorem

An "n-point" correlation function can be rewritten as the sum of all possibe "contractions" of the operators sandwiched between the vaccuum.

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Section 4.4
Feynman Diagrams(Not from the Path Integral)

Wick Theorem Expansion
Of the Scalar n-point function

Power series expansion of the Expansion

Corresponding diagrams

Position Space Feynman Rules

Momenntum Space Feynman Rules

Vacuum Diagrams and the Denominator

Section 4.5
Scattering Cross-sections

The S-Matrix form Wave Packets

The T-matrix and Scattering Amplitudes

Section 4.6

The S-Matrix from Feynman Diagrams

Expanding the final states of S-Matrix Representation

Perturbative expansion and Wick's Theorem

Feynman Rules for Scattering

Section 4.7
Feynman Rules for Fermions

Intro to Dimensional Regularization

Basic Concepts:

- Ultraviolet divergences in Feynman diagrams come from the integration of internal momenta in 4-dim space. We can make integration finite by lowering the dimensionality of the space-time ['t Hooft and Veltman 1972]

$$\int \frac{d^4 q}{(2\pi)^4} \to \int \frac{d^d q}{(2\pi)^d}$$

- Now, Feynman integrals are defined as analytic functions of the space-time dimension d. The ultraviolet divergences manifest themselves as singularities as d goes to 4.

Principles and Axioms:

- Axiom I (linearity): $\int d^d q [a f(q) + b g(q)] = a \int d^d q f(q) + b \int d^d q g(q)$
- Axiom II (scaling): $\int d^d q f(q) = s^{-d} \int d^d Q f(\frac{Q}{s}), \quad s>0$
- Axiom III (tr. invar.): $\int d^d q f(q+k) = \int d^d q f(q)$
- Axiom IV (rot. invar.): $\int d^d q q^\mu f(q^2) = 0$
  $\int d^d q q^\mu q^\nu f(q^2) = g^{\mu\nu}$
- Axiom V (Gauss th.): $\int d^d q \frac{\partial}{\partial q_\mu} F^\mu(q) = 0$

$$\int d^d q \; d^d l \; [...] = \int d^d l \; d^d q \; [...]$$

$$\frac{\partial}{\partial k_\mu} \int d^d q f(k,q) = \int d^d q \frac{\partial}{\partial k_\mu} f(k,q)$$

The Master Integral:

$$I^{(d)}[m^2;\alpha] := \int \frac{d^d q}{(2\pi)^d} \frac{1}{[q^2 - m^2 + i\varepsilon]^\alpha}, \quad \Re[d] < \Re[2\alpha]$$

Can be calculated in the Euclidean space by performing the Wick rotation:

$$I^{(d)}[m^2;\alpha] = i \frac{(-1+i\varepsilon)^{-\alpha}}{(4\pi)^{d/2}} \frac{\Gamma[\alpha - \frac{d}{2}]}{\Gamma[\alpha]} (m^2 - i\varepsilon)^{\frac{d}{2}-\alpha}$$

Loop Integrals Calculations

Feynman Parameters:

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[xA + (1-x)B]^2}$$

$$\frac{1}{A_1 A_2 \cdots A_n} = \int_0^1 dx_1 \cdots dx_n \, \delta(\sum x_i - 1) \frac{(n-1)!}{[x_1 A_1 + x_2 A_2 + \cdots x_n A_n]^n}$$

More Integrals:

$$\int \frac{d^4 q}{(2\pi)^4} \frac{1}{[q^2 - 2q\cdot k - m^2 + i\varepsilon]^\alpha} = I^{(d)}[m^2 + k^2; \alpha]$$

$$\int \frac{d^4 q}{(2\pi)^4} \frac{q^\mu}{[q^2 - 2q\cdot k - m^2 + i\varepsilon]^\alpha} = k^\mu I^{(d)}[m^2 + k^2; \alpha]$$

$$\int \frac{d^4 q}{(2\pi)^4} \frac{q^2}{[q^2 - 2q\cdot k - m^2 + i\varepsilon]^\alpha} = I^{(d)}[m^2; \alpha - 1] + m^2 I^{(d)}[m^2; \alpha]$$

Things to redefine in d-dimentions:

- Minkowski metric: $g_{\mu\nu} g^{\mu\nu} = d$
- Properties of gamma matrices: $\{\gamma^\mu, \gamma^\nu\} = 2 g^{\mu\nu} \Rightarrow \gamma_\mu \gamma^\nu \gamma^\mu = (2-d)\gamma^\nu$
- Trace of gamma matrices: $tr[\gamma^\mu \gamma^\nu] = f(d) g^{\mu\nu}, \quad f(d=4) = 4$
- Fermion and Boson fields ([L]=d): $[\psi] = \frac{d-1}{2}, \quad [\Phi] = [A] = \frac{d-2}{2}$
- E/M and φ^4 Couplings: $[e] = \frac{4-d}{2}, \quad [\lambda] = 4-d$

Following convention is chosen: renormalized couplings have to stay dimensionless in all dimensions. This can be achieved by introducing the arbitrary mass scale μ, so that

$$e_R \to e_R \mu^{\frac{4-d}{2}}, \quad \lambda_R \to \lambda_R \mu^{4-d}$$

Properties of Γ-function:

$$\lim_{\epsilon \to 0} \Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + O(\epsilon), \quad \gamma_E \approx 0.5772, \quad \epsilon = \frac{4-d}{2}$$

$$\Gamma[-n] = (-1)^n \; n! \; \Gamma[\epsilon], \quad n = -1, -2, ...$$

QFT observables are defined on the space of fields. Particles are in constant interaction with this field in the same way as electrons in a solid are in constant interaction with the ions of the grating (real mass vs effective mass). The goal of renormalization is to account for these interactions by mapping unphysical bare parameters into physical observables. $\phi^4 - \text{theory}$ is used as an example.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi_B)^2 - \frac{1}{2}m_B^2\Phi_B^2 - \frac{\lambda_B}{4!}\Phi_B^4$$

Renormalization

Physical Green's function is the renormalized Green's function

$$G_F^{(n)}(x_1,...,x_n) = Z_3^{-n/2}G_B^{(n)}(x_1,...,x_n)$$
$$\Gamma_F^{(n)}(x_1,...,x_n) = Z_3^{n/2}\Gamma_B^{(n)}(x_1,...,x_n)$$
$$\Gamma_F^{(n)}G_F^{(n)} = i\hbar$$

Go from bare to physical parameters:

$$\Phi_B = Z_3^{1/2}\Phi_F$$
$$\lambda_B = (Z_1/Z_3^2)\lambda_F$$
$$m_F^2 = (Z_3/Z_0)m_B^2$$

Get the renormalized Lagrangian

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\Phi_F)^2 - \tfrac{1}{2}m_F^2\Phi_F^2 - \tfrac{\lambda_F}{4!}\Phi_F^4 +$$
$$+\tfrac{1}{2}(Z_3-1)(\partial_\mu\Phi_F)^2 - \tfrac{1}{2}(Z_0-1)m_F^2\Phi_F^2 - (Z_1-1)(\lambda_F/4!)\Phi_F^4$$

Renormalization conterterms

How to construct counterterms such that the divergent contributions coming from the first three terms exactly cancel out that counterterms?

Regularization

What kind of infinities and how many of them do we have in the theory? Do we have enough bare parameters (we need one per infinity)?

Taylor expansion of the diverging tems yields renormalization conditions which fix conterterms:

$$\tilde{\Gamma}_F^{(4)}|_\mu = \lambda_F$$
$$\frac{\partial}{\partial p^2}\tilde{\Gamma}_F^{(2)}(p)|_{p^2=m_F^2} = 1$$
$$\sum\nolimits_F(p)|_{p^2=m_F^2} = 0$$

The number of the equations must be equal to the number of renormalization constants

How unique is the regularization procedure?

"...once we have specified the counterterms, which cancel the infinities, we can make a finite changes in them" (renormalization group)

Momentum power counting (structure of the divergences)

We can limit ourselves with one particle irreducible diagrams (1PI) only:
- include all the possible divergent loops
- simplifies identification of necessary Lagrangian counterterms

$D = 4L - N$ - Superficial degree of divergence

$L = I - V + 1$ - Number of independent loops

I - Number of internal lines

V - Number of vertices

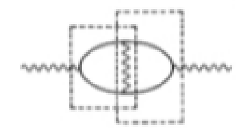
N - Number of external lines

Divergent diagrams satisfy: $D \geq 0$

Extracting the counterterms

In this case we
- Identify all the divergent subdiagrams in the process
- Replace them with equivalent non-diverging part plus the conterterm
- Repeat the procedure for higher order terms until all the divergencies are elliminated.

Possible pitfall: overlaping divergencies



Dimensional regularization

- Go to $d - \epsilon$ dimensions
- Use the Feynman trick to merge the denominator
- Apply Wick rotation and calculate the integral $\ell_0 \to i\ell_0$
- Isolate the divergent terms

Weinberg Theorem:

Feynman integral converges if the degree of divergence of the diagram as well as the degree of divergence associated with each possible subintegration over loop momenta is negative.

Axioms of renormalization:

- Physical theories are renormalizable theories (?)
- Bare (unrenormalized) parameters of the theory are not observable
- The resulting observable parameters are independent of renormalization scheme (renormalization group)

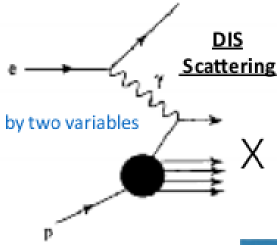
Experimentally

$$\frac{d^2\sigma}{d\Omega\, dE'} = \frac{\alpha^2}{q^4}\left(\frac{E'}{E}\right) l^{\mu\nu} W_{\mu\nu}$$ ← Unknown Hadronic Tensor

$$\frac{d\sigma}{dq^2 d\nu} = \frac{4\pi\alpha^2}{q^4}\frac{E'}{E}\left[W_2\cos^2\left(\frac{\theta}{2}\right) + 2W_1\sin^2\left(\frac{\theta}{2}\right)\right]$$ The structure functions depend by two variables

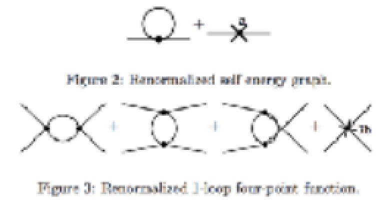
Observed dependence only by x Bjorken at high Q^2

Parton Model: If Q^2 is high, I am scattering with a point-like constituent carrying x fraction of the total momentum of the proton (in collinear approximation).



The QCD at high energy describes free particles ?

Theoretically Counter terms in the Lagrangian



Different Renormalisation Schemes (MS, etc.)

? **The physical quantities cannot depend by the renormalisation scheme**

The renormalisation group

Two renormalisation scheme: R, R' :

$$\Gamma_R = Z(R)\Gamma_0 \qquad \Gamma_{R'} = Z(R')\Gamma_0$$

→

$$\Gamma_{R'} = Z(R', R)\Gamma_R$$
$$Z(R', R) \equiv Z(R')/Z(R)$$

→ **Group multiplication law** $Z(R'', R')Z(R', R) = Z(R'', R)$; **Identity:** $Z(R, R) = 1$

$$\Gamma_0^{(n)}(p_i, g_0, m_0) = Z_\phi^{-n/2}\Gamma^{(n)}(p_i, g, m, \mu)$$ renormalised at point μ

→ dimensionless derivative $\mu(d/d\mu)$.

$$0 = \mu\frac{\partial}{\partial\mu}\Gamma_0^{(n)} = \left(\mu\frac{\partial}{\partial\mu}Z_\phi^{-n/2}\right)\Gamma^{(n)} + Z_\phi^{-n/2}\left(\mu\frac{\partial}{\partial\mu}\Gamma^{(n)}\right)$$

$$\frac{d}{d\mu} = \frac{\partial}{\partial\mu} + \frac{\partial g}{\partial\mu}\frac{\partial}{\partial g} + \frac{\partial m}{\partial\mu}\frac{\partial}{\partial m}$$

Redefinition of quantities

$$\beta(g) \equiv \mu\frac{\partial g}{\partial\mu} \quad [1]$$
$$\gamma(g) \equiv \mu\frac{\partial}{\partial\mu}\log\sqrt{Z_\phi}$$
$$m\gamma_m(g) \equiv \mu\frac{\partial m}{\partial\mu}$$

The vertex function changes, as function of the normalisation point such as:

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - n\gamma(g) + m\gamma_m(g)\frac{\partial}{\partial m}\right)\Gamma^{(n)}(p_i, g, m, \mu) = 0$$

[1] Integrated and Taylor expand: $$g^{n-1}(\mu) = \frac{g(\mu_0)^{n-1}}{1-(n-1)bg(\mu_0)^{n-1}\log(\mu/\mu_0)}$$

a scale of momentum $p_i \to e^t p_i = \lambda p_i$

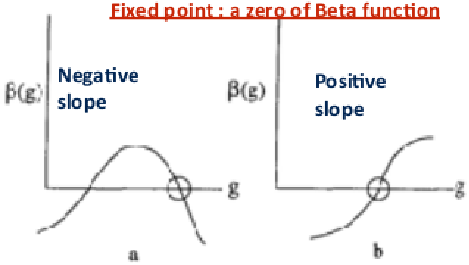
$$\left(-\frac{\partial}{\partial t} + \beta(g)\frac{\partial}{\partial g} + [D + n\gamma(g)]\right)\Gamma^{(n)}(\lambda p_i, g) = 0$$

This dimension of the vertex function, from Lorentz invariance.

We can solve it →

$$\Gamma(\lambda p_i, g) = \Gamma(p_i, \bar{g})\exp\left(\int_g^{\bar{g}} dg' \frac{\bar{\gamma}(g')}{\beta(g')}\right)$$ **By introducing a running coupling constant:** $$\frac{d\bar{g}(g,t)}{dt} = \beta(\bar{g})$$

Fixed point : a zero of Beta function



$$\begin{cases} \beta'(g_F) < 0 : & \text{Ultraviolet stable} \\ \beta'(g_F) > 0 : & \text{Infrared stable} \end{cases}$$

Taylor expansion around a fixed point

$$\beta = \mu\frac{\partial}{\partial\mu}g = (g - g_F)\beta'(g_F) + \cdots$$ *small*

QED

$$e^2(\mu) = \frac{e^2(\mu_0)}{1 - (e^2(\mu_0)/6\pi^2)\log\frac{\mu}{\mu_0}}$$

ϕ^4

$$g(\mu) = \frac{g_0(\mu_0)}{1 - (3/16\pi^2)g_0(\mu_0)\log(\mu/\mu_0)}$$

Asymptotic Freedom

Gauge theory: $$g_0 = g\mu^{\epsilon/2}\frac{Z_1}{Z_2\sqrt{Z_3}} = g\mu^{\epsilon/2}\left[1 - \frac{g^2}{8\pi^2\epsilon}\left(\frac{11}{6}C_{ad} - \frac{2}{3}C_f\right)\right] + \cdots$$

→ Solving for β

$$\beta = \mu\frac{\partial g}{\partial\mu} = -\frac{g^3}{16\pi^2}\left(\frac{11}{3}C_{ad} - \frac{4}{3}C_f\right) + \cdots$$

$$\frac{11}{3}C_{ad} > \frac{4}{3}C_f$$

Asymptotic freedom