Lattice 2025 satellite school

lecturer: maxim.mai@faculty.unibe.ch
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I Introduction

QCD: theory of strong interaction

- fundamental force of nature
- strongest among others: SI 1, EM 10^{-2}, WI 10^{-13}, "Gravity" 10^{-38}

6 spin 1/2 matter fields: 3 light + 3 heavy compared to $\Lambda_{QCD} \sim 1$ GeV
gauge $(SU(3)_c)$ bosons are gluons

$\uparrow$ non-abelian group
$\uparrow$ gluon self interaction 
$\uparrow$ confinement: only colorless objects can be observed.

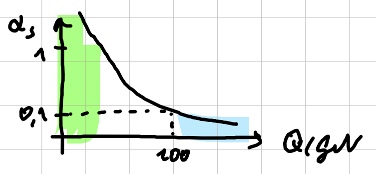
HADRONS
- $q\bar{q}$ (color/anticolor) → mesons
- qqq (rgb) → baryons

} „naive" quark model — why does it work so well? S. Weinberg

- glueballs
- hybrids
- pentaquarks (mes + bar?)
- tetraquarks

} → lots of ongoing research: LQCD continuum

interaction $\alpha_s = \frac{g_s^2}{4\pi}$

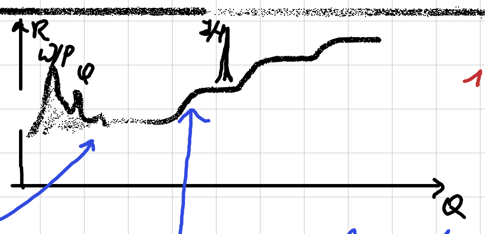


- large momentum exchange ↔ short distance, $\alpha_s \ll 1$

⇒ PQCD (perturbative QCD) calculations become relevant

Example: R-ratio

$$R = \frac{\sigma_{e^-e^+ \to q\bar{q}+X}}{\sigma_{e^+e^- \to \mu^+\mu^-}}$$

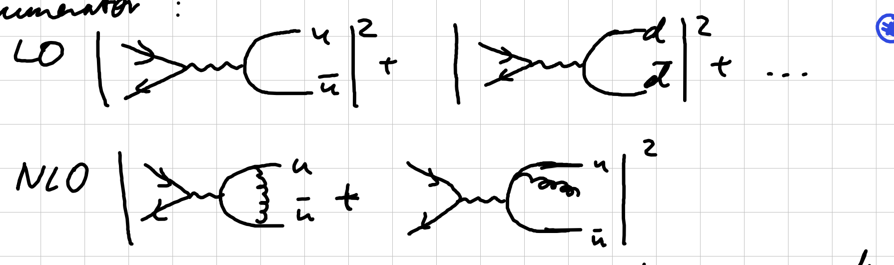


1707.01044

resonances ω, φ, ρ …

new types of quarks can produce on-shell contributions. ⊛

denominator has no QCD effects beyond renormalization.
numerator: ⊛



X can have more jets in the calorimeter

⋮

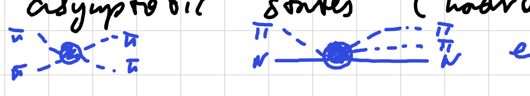
N^4LO: 10k diagrams [currents status]

LO + NLO: dim reg.: $R = N_c \cdot \sum_i Q_i^2 \left(1 + \frac{\alpha}{\pi}\right)$ ← corrections.

this lecture →

- small momentum exchange ↔ long distances $\alpha_s \gg 1$

↳ objects with strong charge (color) cannot leave asymptotic states (hadrons) ⟷ confinement.



↳ there are $\mathcal{O}(400)$ mesons, $\mathcal{O}(50)$ baryons — p, n, π, π(1300), ρ, ω … PDG

↳ most of them are excited states. States decaying to other hadrons, e.g.

$$N^*(1440) \to N\pi,\ N\pi\pi, \dots$$

$$\omega(782) \to \rho\pi \to \pi\pi\pi$$

⋮

⇒ how does one access such states (theory / experiment)

↳ we cannot use PQCD – there is no ordering as $\alpha_s \approx 1$ and
↳ we would need to calculate all. ↯

SOLUTIONS FROM QCD

《systematically improvable》

Lattice QCD | **Effective field theories (ChPT)**

⟵ S-matrix theory ⟶

Review: 2206.01477

REMINDER

S-matrix, Green's function, Generating functional, ...

- Green's function $\hat{=}$ Correlation function

$\langle \Omega | T \, \varphi(x) \, \varphi(y) | \Omega \rangle$

$\uparrow$ time ordering $\uparrow$ ground state of the theory

for free theory this is simply Feynman propagator $D(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i \, e^{-ip(x-y)}}{p^2-m^2+i\varepsilon}$

↳ Wick's theorem turns this into all contractions and thus relates it to <u>Feynman diagrams</u> (propagators, loops ...)

- How to connect this to measurable quantities?

$_{out}\langle p_1' \dots | p_1, \dots \rangle_{in} =: \langle p_1' \dots | \hat{S} | p_1 \dots \rangle$ S-matrix

↳ $\hat{S} = \underbrace{\mathbb{1}}_{\text{no interaction}} + \underbrace{i}_{\text{convention!}} \hat{T}$

invariant matrix element $\mathcal{M}$:

$$\langle p_1' \dots | \hat{T} | p_1 \dots \rangle = \underbrace{(2\pi)^4 \cdot \delta^4(P-P')}_{\text{sometimes dropped}} \, i \mathcal{M}(p_1 \dots \to p_1' \dots)$$

cross sections

$$d\sigma = \frac{1}{2E_A 2E_B |v_A - v_B|} \cdot \underbrace{\prod_f \frac{d^3 p_f}{(2\pi)^3} \cdot \frac{1}{2E_f}}_{\text{LIPS}} \times \left| \mathcal{M}_{p_A p_B \to \{p_f\}} \right|^2 (2\pi)^4 \delta^4(p_A + p_B - p_f)$$

$$\frac{d\sigma}{d\Omega} = \frac{|\mathcal{M}|^2}{64\pi^2 E_{cm}^2} \quad \text{(identical particles)}$$

- How to connect S-matrix & Green's functions / Feynman diagrams

LSZ reduction formula

$$\int d^4x_1 \, e^{ip x_1} \dots \langle \Omega | T \, \varphi(x_1) \dots | \Omega \rangle = \underbrace{\frac{i\sqrt{Z}}{p_1^2 - m^2 + i\varepsilon} \dots}_{\text{external propagators and WF-renormalization}} \underbrace{\langle p_1' \dots | \hat{S} | p_1 \dots \rangle}_{\text{computated part}}$$

↳ observation each particle – antiparticle on the LHS is transformed by $p_i^0 \to -p_i^0$. This transforms particle $\varphi(\vec{p}) \to \varphi^*(\vec{p}) \Rightarrow$ <u>crossing symmetry</u>

- Path-integral / Generating functional / ...

$$\underset{\substack{\uparrow \\ \text{source}}}{Z(J)} = \int \mathcal{D}\varphi \, \exp\left[i \int d^4x \left(\mathcal{L} + J(x) \, \varphi(x) \right) \right]$$

$$\langle \Omega | T \varphi \dots | \Omega \rangle = \frac{1}{Z|_{J=0}} \cdot \left(\frac{-i \, \partial}{\partial J(x_1)} \right) \dots Z(J) \Big|_{J=0}$$

$$= \frac{\int \mathcal{D}\varphi \; \varphi(x_1) \dots \exp\left[i \int d^4x \, \mathcal{L} \right]}{\int \mathcal{D}\varphi \; \exp\left[i \int d^4x \, \mathcal{L} \right]}$$

Effective field theories / Chiral perturbation theory (ChPT)

2502.02654 / hep-ph/0505265

- QCD Lagrangian $m_{uds} \ll m_{c+b}$

$$\mathcal{L}_{QCD} = \sum_f \bar{q}_f^a \left(i\gamma_\mu D^\mu_{ab} - m_f \delta_{ab}\right) q_f^b - \frac{1}{4} \mathcal{G}^a_{\mu\nu} \mathcal{G}_a^{\mu\nu}$$

↓ light quarks $m_\ell \to 0$ (chiral limit)

$$\mathcal{L}_0 = \sum_{f=uds} \left(\bar{q}_L^f \, i\gamma_\mu D^\mu q_L^f\right) + \left(\bar{q}_R^f \, i\gamma_\mu D^\mu q_R^f\right) - \frac{1}{4} \mathcal{G}^a_{\mu\nu} \mathcal{G}_a^{\mu\nu}$$

$$q_{L/R} = \frac{1}{2}(1 \mp \gamma_5)\, q$$

(chiral symmetry) $U_R(3) \times U_L(3) = U_V(1) \times U_A(1) \times SU_A(3) \times SU_V(3)$

(B#) ✓   Quantum broken × $\boxed{\text{Phys Rev 177}}$

⊗ cannot be broken Wafa Witten NPB 234 (1984)

⊗ broken hidden symmetry otherwise there would be a parity doubler for each state of Hadron Sp.

$$SU_A(3) \times SU_V(3) \xrightarrow{SSB} SU_V(3)$$

Goldstone theorem
each generator → GB

$\{3\pi, 4K, 1\eta\}$ ⟷ indeed very light (massless in the chiral limit)

- This was foundation of PCAC studies
- Chiral perturbation theory : $\mathcal{L}_{EFF}(\varphi_{\uparrow}, s, p, v, a)$ sources

↳ formulated in asymptotically stable DOF^{GB} (Mesons & Baryons)

↳ $$Z_{eff} = \int \mathcal{D}\Phi_{eff} \, \exp\left[i \int d^4x \, \mathcal{L}_{EFF}[\Phi, v, s, p, a]\right]$$

a) Weinberg conjecture : as long as theory contains most general form and is consistent with all symmetries it provides the same physics result PA 96 (1979) 327-340

b) $\mathcal{L}_{ChPT} = \mathcal{L}^{(2)}_\varphi + \mathcal{L}^{(4)}_\varphi + \ldots$ mesons NPB 250 (1985) 465-516

$\mathcal{L}_{BChPT} = \mathcal{L}^{(1)}_{MB} + \mathcal{L}^{(2)}_{MB} + \ldots$ baryon-meson

there are ∞ many terms but power counting exists.

↳ renormalization order-by-order

↳ predictive power

↳ coefficients are called low-energy constants
not known from theory! need input

- Examples of calculations

$$\mathcal{L}^{(2)}_\varphi = \frac{F^2}{4} \langle D_\mu u \, D^\mu u^+ + \chi^+ u + \chi u^+ \rangle$$

$$u = \exp\left(\frac{i}{F\sqrt{2}} \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\eta}{\sqrt{6}} \end{pmatrix}\right), \qquad \chi \sim \begin{pmatrix} m_u & & \\ & m_d & \\ & & m_s \end{pmatrix}$$

$$\Rightarrow \quad M_\varphi^2 = \sum_{\ell = uds} a \cdot m_\ell$$ indeed disappears in χlimit

$$\Rightarrow \quad \underbrace{T^{I=0}_{\varphi\varphi \to \varphi\varphi}}_{\text{T-matrix}} = \frac{2s - M_\pi^2}{F^2} + \frac{1}{F^4}\left(P(s,t,u)^{\ell_1 \ldots \ell_4} + \text{loops}\right) + \ldots$$

low energy constants
unknown ← $\boxed{\text{LQCD}}$

$\boxed{\text{LQCD}}$ ← $m_u/m_d/m_s$ independent

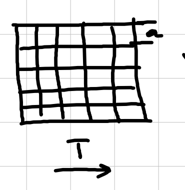
QFT on the lattice {bird's view}

- Path integral formulation $\longrightarrow$ Euclidean discretized space-time

Otherwise the integrand of Path Integral oscillates

$$Z[J] = \int \mathcal{D}\varphi \exp\left[-S[\varphi] + \int d^4x \, J\varphi\right]$$

finite computational times



$\times L$ (3 times)

$L = N \cdot a$

- Discretized version of derivative $\frac{1}{a}\left(\varphi(\vec{x} + a\vec{\mu}) - \varphi(x)\right)$ etc. do require us to say what happens at the lattice boundary.

$\Rightarrow$ Boundary condition $\varphi(\vec{x} + Na\vec{\mu}) = e^{i\theta}\varphi(x)$

twisting

$\theta = 0$ for PBC

$\Rightarrow$ FT to momentum space : $\vec{p} = \frac{2\pi}{L} \cdot \vec{n}$ $\vec{n} \in \mathbb{Z}^3$

$p_{max} = \frac{\pi}{a}$

same #dof in CS & MS

↳ still very large $p_{max} = \frac{\pi \cdot N}{m L} \approx N$

Spectrum : asymptotic states contain $\begin{cases} \text{vaccuum } |0\rangle \\ |\pi^+(p)\rangle, |N(\vec{p})\rangle, \dots |\pi\pi\pi\rangle \dots \end{cases}$

↳ correlation function

$$\langle 0 | T \varphi_{\pi^+}(x) \, \varphi^+_{\pi^+}(y) | 0 \rangle = \frac{\int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}U \, \varphi_{\pi^+}(x) \, \varphi^+_{\pi^+}(y) \, e^{-S[\psi, \bar{\psi}, U]}}{\int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}U \, e^{-S(\psi \bar{\psi} U)}}$$

way to calculate

see other lectures

$\bar{d}(x)\gamma_5 u(x)$

appropriate operators.

Srijit TALK

$$= A \cdot e^{-E_1 \cdot x_4} + \dots$$

large x_4 behaviour

Energy eigenvalues

$\Rightarrow$ Extract E_1 (plateaus $x_4 \to \infty$)

$\Rightarrow$ if multiple operators are included GEVP.

(details in other lectures)

Summary :

new information unphysical quark masses etc...

$\oplus$ QCD degrees of freedom

correlators

$\oplus$ non-perturbative calculations (Path integral) of Greens' functions.

$\ominus$ discretized space-time ($a > 0$) continuum limit is necessary.

$\ominus$ Euclidean metric (needed to have well-defined weight in Z

$\ominus$ Finite Volume $\to$ BC $\to$ discretized momenta & spectrum

$$G = \frac{1}{E - H} = \sum \frac{|n\rangle\langle n|}{E - E_n} \longleftrightarrow \langle p_1 \dots | S | p_1' \dots \rangle$$

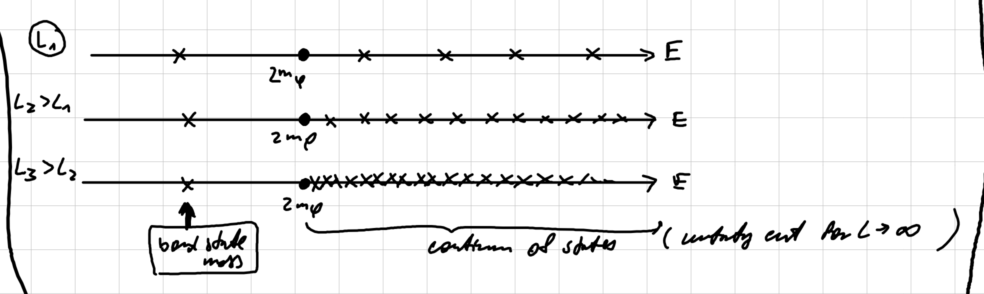
discrete

LSZ relation to

Lattice QCD

input from continuum theories e.g. ChPT

Suppose we have $\varphi\varphi \to \varphi\varphi$ builds a bound state.

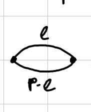


- Limits to access continuum QFT/physics :

  $a \to 0$, $L = N \cdot a = \text{const}$ ($N \to \infty$)

  continuum physics in finite volume

  ↳ for example this means that ($T \gg L$, c.f. plateaus)



$$: \quad I(p) = \frac{1}{L^3} \cdot \int \frac{dl_4}{(2\pi)} \cdot \sum_{\vec{l} \in \frac{2\pi}{L}\mathbb{Z}^3} \frac{1}{(m^2 + l^2)} \cdot \frac{1}{(m^2 + (p-l)^2)}$$

Arnab Sen

„delicate series of limits)

Summary #2

Solution #1 (LQCD)	Solution #2 (ChPT/EFT)
• quarks & gluons • discretized ($a > 0$) • Euclidean ($t \mapsto it$) • finite volume grid	• Effective degrees of freedom ($\pi, K \dots$) • Weinberg conjecture : if symmetries are respected one has same content physically. • heavy DOF are integrated out but coefficients remain $\mathcal{L}_{QCD} \longmapsto \mathcal{L}_{EFT} = \sum a_i \, \mathcal{O}_i$ low energy constants.

non-perturbative input

quark mass dependence

analytical tools & benchmarks

$\Longleftarrow$ S-matrix theory $\Longrightarrow$

BRIDGE

Example #1 : two-body scattering

- spinless φ, mass m

$\boxed{\begin{array}{l}2206.01477\\2502.02654\end{array}}$

- S-matrix

$$\underset{(t=-\infty)}{|\varphi\varphi\rangle_{in}} \xrightarrow{\hat{S}} \underset{(t=+\infty)}{|\varphi\varphi\rangle_{out}}$$

↳ keeping only connected part: $\hat{S} = \mathbb{1} + i\hat{T}$ convention

↳ general properties:

a) crossing symmetry.
particle - antiparticle ⇒ analytic conversion exist between e.g.

$\pi N \to \pi N$ & $\pi\pi \to N\bar{N}$ etc...

b) Analyticity.
Rooted in causality requirement
S-matrix is analytic function in all inner products of momenta promoted $\underline{\mathbb{C}}$.
$(\vec{p}_1\vec{p}_2), \dots$

↳ how many (angular momentum, energy ... onshellness)

$\# dof = 3(n+m) - 10$

c) Unitarity
probability to measure anything is $P=1$ $(SS^\dagger = \mathbb{1})$

$1 = \sum |\langle\ |S|m\rangle|^2 = \sum_m \langle\ |S|m\rangle\langle m|S^\dagger|\ \rangle = \sum_{nn'} a_n a^*_{n'} \langle n|SS^\dagger|n\rangle$

this has to hold for any a_n $a_{n'}$ ⇒ $\underline{\underline{SS^\dagger = \mathbb{1}}}$

↳ $\hat{T} - \hat{T}^\dagger = i\,\hat{T}\cdot\hat{T}^\dagger$ (imaginary part is fixed)

We were interested in matrix elements of $\hat{T}$ (LSZ → correlators)

$$\langle\varphi\varphi|T-T^\dagger|\varphi\varphi\rangle = \sum_{n=2}^{N} \int \frac{d^4k}{(2\pi)^4}\cdots\frac{d^4k_n}{(2\pi)^4} (2\pi)\delta^+(k_1^2+m^2)\dots(2\pi)^4\delta^4(P-k_1\dots)\times$$
$$\times \underbrace{\langle\varphi\varphi|T|k_1\dots k_n\rangle\langle k_1\dots k_n|T^\dagger|\varphi\varphi\rangle}$$

- intermediate onshell states
- how many (N?)
  - ↳ allowed by QN
  - ↳ $N\cdot m < \sqrt{s}$ ⇒

RANGE OF APPLICABIL	
two body	$\sqrt{s} < 3m$
three body	$\sqrt{s} < 4m$
4-body	$\sqrt{s} < 5m$
⋮	

$\sqrt{s}/m < 3$

$$RHS = \int \frac{d^4k}{(2\pi)^4}(2\pi)\delta^+(k^2-m^2)(2\pi)\delta^+((P-k)^2-m^2)\langle\varphi\varphi|T|k\,P\text{-}k\rangle\langle P\text{-}k,k|T^\dagger|\varphi\varphi\rangle$$

↳ free variables s & $\cos\theta = z$ (for example)
↳ apply $\int d\Omega\, Y_{lm}(\Omega) \to \quad \leftarrow \int d\Omega\, Y_{lm}(\Omega)$ on RHS=LHS

↳ $\text{Disc}\, T_{lm}^{-1}(s) = -\frac{p_{cm}}{8\pi\sqrt{s}}$ $\qquad p_{cm} = \sqrt{\frac{s}{4}-m^2}$

K-matrix form

$$T_{lm} = \frac{8\pi\sqrt{s}}{p_{cm}\cot\delta - ip_{cm}}$$

Self-energy integral (SE)
⟺ Dispersion relation
⟺ Lippmann Schwinger Eq.

$$T_{lm} = -\frac{1}{2}\cdot\left(\tilde{K}^{-1} - \Sigma^{IV}\right)^{-1}$$

$$\int\frac{d^3k}{(2\pi)^3}\cdot\frac{1}{2E_k}\cdot\frac{1}{s-4E_k^2+i\varepsilon}$$

$$\text{Im}\,\Sigma^{IV} = -\frac{p_{cm}}{16\pi\sqrt{s}}$$

Shifting Real parts

$$\frac{p\cot\delta}{8\pi\sqrt{s}} - \frac{ip_{cm}}{8\pi\sqrt{s}} = -2\tilde{K}^{-1} + \text{Re}\,\Sigma^{IV}\,2 - \frac{ip_{cm}}{8\pi\sqrt{s}}$$

$$\boxed{\tilde{K}^{-1} = -\frac{p\cot\delta}{16\pi\sqrt{s}} + \text{Re}\,\Sigma^{IV}}$$

- 1709.08222
- review above

- What happens when momenta are discretized?

$$\Sigma^{IV} \longrightarrow \Sigma^{FV} = \frac{1}{L^3}\sum_{k\in\frac{2\pi}{L}\mathbb{Z}^3}\frac{1}{2E_k}\frac{1}{s-4E_k^2}$$

- $\Sigma^{FV} \in \mathbb{R}$
- $\Sigma^{FV} = \infty$ ⟺ $|\varphi\varphi\rangle$ is onshell ⟺ $\underline{\underline{\sqrt{s} = 2\sqrt{k^2+m^2}}}$

↳ if Σ^{FV} regular $(s < 2m)$ Regular summation theorem applies

$$\left(\int - \Sigma\right)\frac{1}{2E_k}\cdot\frac{1}{s-4E_k^2} = e^{-ML}$$

usually ML is large enough such that this can be neglected.

↳ same holds for $\tilde{K}^{-1} \in \mathbb{R}$ which is volume-indep. besides e^{-ML} effects.

⇒ two body quantization condition:

$\sqrt{s}$ is a predicted energy eigenvalue iff

$$T^{FV}(s) = \frac{1}{\tilde{K}^{-1} - \Sigma^{FV}} = \infty$$

$$\Leftrightarrow \tilde{K}^{-1} = \Sigma^{FV}$$

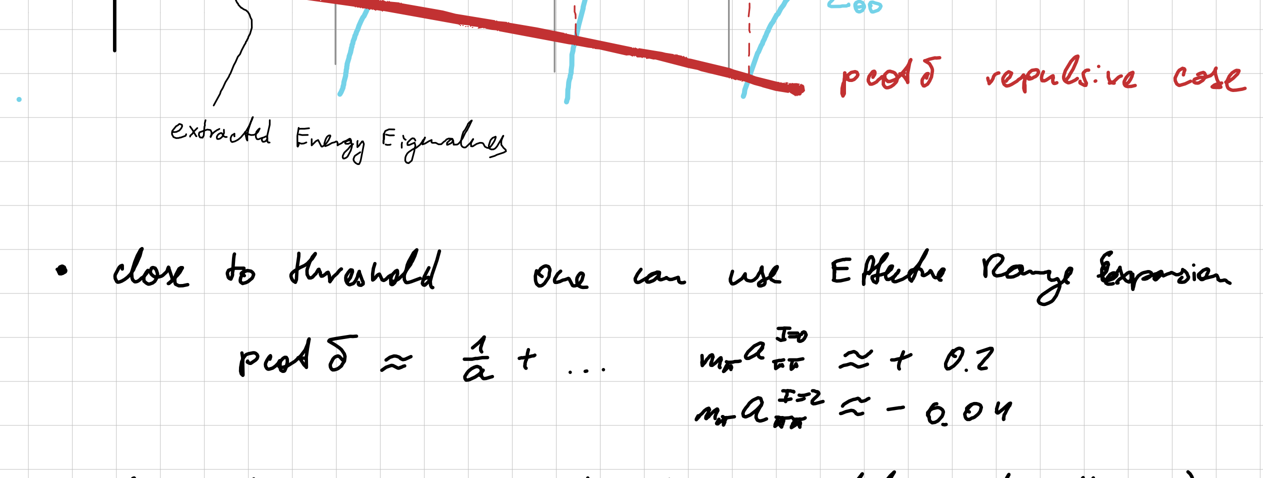
$$\Leftrightarrow p_{cm}\cot\delta = \underbrace{\left(-\Sigma^{FV} + \text{Re}\,\Sigma^{IV}\right)16\pi\sqrt{s}}_{Z_{00}+e^{-ML}\ \text{(Lüscher Zeta function)}}$$

$$\frac{1}{2E_k}\cdot\frac{1}{s-4E_k^2} = \frac{1}{4\sqrt{s}}\frac{1}{p_{cm}^2-k^2} - \underbrace{\frac{1}{4E_k^2}\cdot\frac{1}{\sqrt{s}+2E_k} + \frac{1}{4E_k^2}\cdot\frac{1}{\sqrt{s}}}_{\text{regular}}$$

$$\frac{1}{L^3}\sum_q\frac{1}{p_{cm}^2-q^2} - \frac{q_{max}}{2\pi^2}\xrightarrow{\lim q_{max}\to\infty} -\frac{1}{2\pi^{3/2}L}Z_{00}$$

→ 1107.3988
⟺ Lüscher QC.

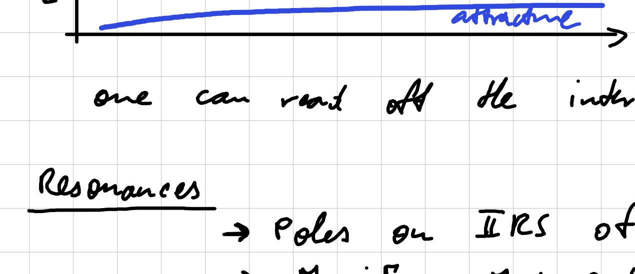
↳ practical example



- close to threshold one can use Effective Range Expansion

$$p\cot\delta \approx \frac{1}{a} + \dots \qquad m_\pi a_{\pi\pi}^{I=0} \approx +0.2 \qquad m_\pi a_{\pi\pi}^{I=2} \approx -0.04$$

- Volume dependence (repeat LQCD simulation at other L)

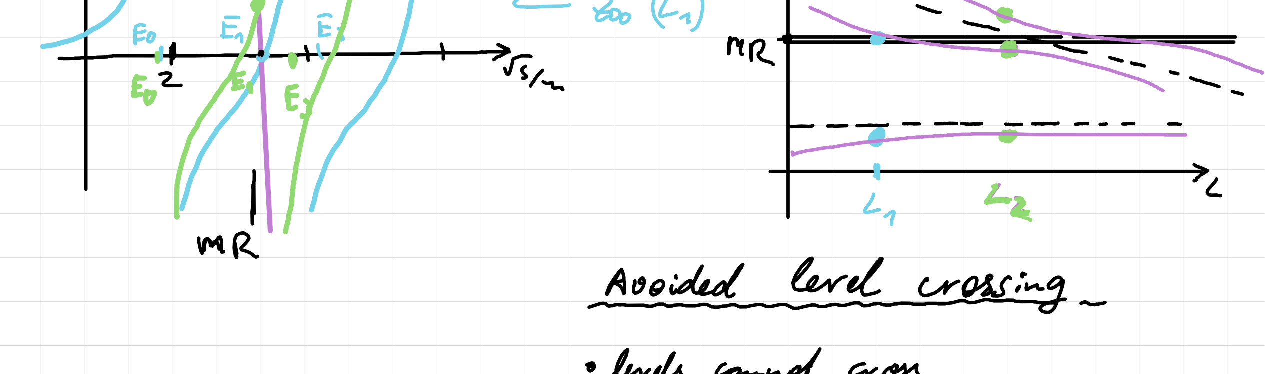


$Z_{00} = \infty$ non-interacting level

one can read off the interaction from such a plot.

- Resonances
  - → Poles on IIRS of T-matrix $s \in \mathbb{C}$
  - → $M - i\Gamma/2$ Mass and width for narrow $(\Gamma \approx 0)$ states.

suppose $\Gamma \approx 0$ FV-spectrum



Avoided level crossing

- levels cannot cross
- but probability to have a level (L) is higher at $\underline{\underline{m_R}}$.

- ChPT input:

↳ $p\cot\delta$ can be obtained from Feynman diagrams

↳ pion mass dependence is dictated by ChPT
↳ Volume dependence is dictated by onshell condition

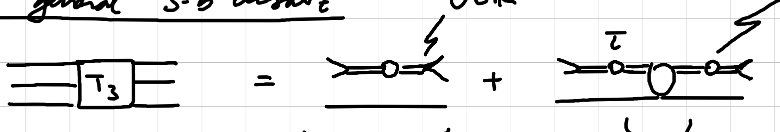
Example #2 : 3body scattering

- Onshell-states → Discontinuity of T-matrix
  → Non-intuitive FV levels.

  Real part → volume independent, physical interaction.

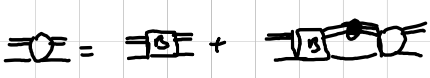
$\Rightarrow$ separate on/off-shell states ! (advantage of Diagram. approach)

- Most general 3-b ansatz



new in 3-body case

general Bethe Salpeter ansatz



integral equation.

B & τ are unknowns

↳ unitarity condition $3 < \frac{\sqrt{s}}{m} < 4$

$$\langle\varphi\varphi\varphi| T - T^\dagger |\varphi\varphi\varphi\rangle = i\int \frac{d^4k_1}{(2\pi)^4}\cdot(2\pi)\,\delta^+(k_1^2-m^2)\ldots\langle\varphi\varphi\varphi|T|k_1k_2k_3\rangle\langle k_1k_2k_3|T^\dagger|\varphi\varphi\varphi\rangle$$

$T_c + T_d$ $\quad$ $T_c^\dagger + T_d^\dagger$

$\# = 4$

↳ combinatorics of intermediate states : $|k_1k_2k_3\rangle\langle k_1k_2k_3|$

2+1 cut. $\quad$ or $\quad$ 3 cut $\quad$ $\#=2$

$\Rightarrow$ $\# = 8$ total

plug in RHS = LHS (1709.08222)

$$\Rightarrow \text{disc}\,\tau^{-1} = -i\frac{p_{cm}}{4\pi\sqrt{\sigma}}$$

2b inv. mass^2

$$\Rightarrow \tau = \frac{1}{\tilde{K}(\sigma)^{-1} - \Sigma^{FV}(\sigma)}$$

$$\text{disc}\,B = 2\pi i\,\frac{\delta\left(Q^0 - \sqrt{m^2+\vec{Q}^2}\right)}{2\sqrt{Q^2+m^2}}$$

$$\frac{1}{E_x(\sqrt{s}-E_p\cdot E_p\cdot -E_x)}$$

$$B = \frac{1}{u^2-m^2+i\varepsilon} + C$$

real term
regulator/vol. indep
3-body force

3-body Scattering Amplitude (IVU)

$$\boxed{T_3 = \vartheta\cdot\tau\,\vartheta + \vartheta\tau\,T\,\tau\vartheta}$$

$$\hookrightarrow \boxed{T = (B+C) + \int\frac{d^4l}{(2\pi)^4}\cdot(B+C)\,\tau\cdot T}$$

3body Quantization Condition (FVU) (related approaches NREFT/RFT)

replace $\int\frac{d^3l}{(2\pi)^3} \longmapsto \frac{1}{L^3}\sum_l$ & $\Sigma^{IV} \to \Sigma^{FV}$

$$\sqrt{s} \in \text{EEV} \iff T_3^{FV}(s) = \infty$$

$$\iff E_L^{*}\,\vartheta\,\underbrace{\frac{1}{\tilde{K}^{-1}-\Sigma^{FV}}}_{\tau}\,\vartheta + \vartheta\tau\left[\mathbb{1} - E_L^{-1}BC\tau\right]^{-1}\cdot BC\tau\vartheta = \infty$$

$*$: $\delta(\vec{p}-\vec{q})\cdot 2E_l \overset{FV}{\Rightarrow} L^3\delta_{pq}\cdot 2E_l$

$$\iff E_L\tau + E_L\left[E_L\tau^{-1} - BC\right]^{-1}BC\,\tau = \infty$$

$$\iff E_L\left[E_L\tau^{-1}-BC\right]^{-1}\left(E_L - \cancel{BC}\tau + \cancel{BC}\tau\right) = \infty$$

$$\iff \boxed{\det\left[B+C - E_L\left(\tilde{K}(\sigma)^{-1} - \Sigma^{FV}(\sigma)\right)\right] = 0}$$

1709.08222

- applications :

  $\pi\pi$
  $\bar{K}N$
  πN
  ⋮

  $\pi\pi\pi$ $I=3$ repulsive but proof of concept 1807.04746

  $\pi\pi\pi$ $I=0$ ⇝ $\omega(782)$ 2407.16659

  $\pi\pi\pi$ $I=1$ ⇝ $\pi(1300)$ 2510.09476

- beyond scattering : Lüscher-Lellouch form factor calculation

$$|F(t)|^2 = \left|L^{3/2}\langle E_n|\,j(0)\,|0\rangle\right|^2\frac{2\pi}{p^2}E_n^2\left|\delta'(p) + \varphi'(p_n)\right|$$

smooth

determined from Lattice irregular $\quad\times\quad$ phase-shift 2b. scattg $\quad$ Lüscher Zoo $\quad$ irregular

if also scattering determined then $|F(t)|^2$ & phase$(F(t))$ can be determined.