
MAXIM MAI

UNIVERSITY OF BERN

THE GEORGE WASHINGTON UNIVERSITY

Seminar PSI
26.03.2026

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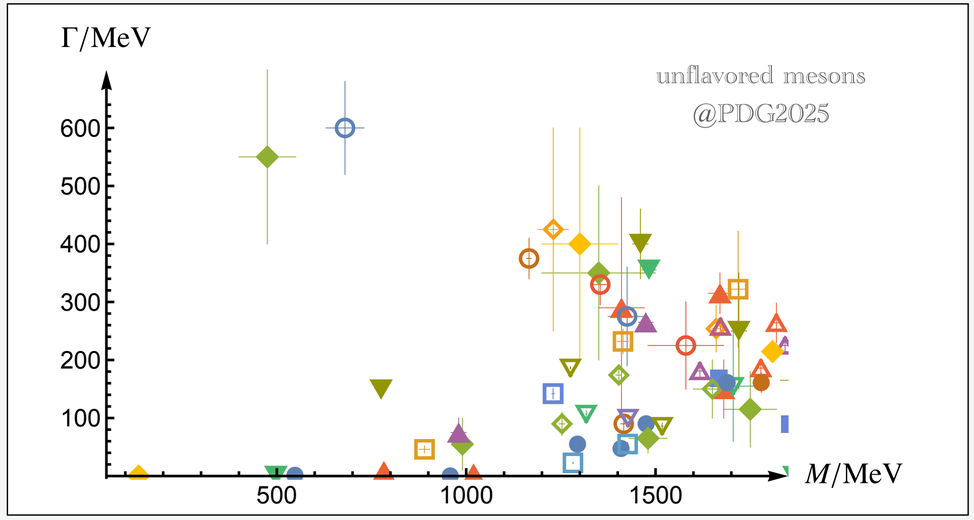
DOE: DE-SC0016582, DE-AC05-06OR23177, DE-FG02-95ER40907
NSF: PHY-2012289

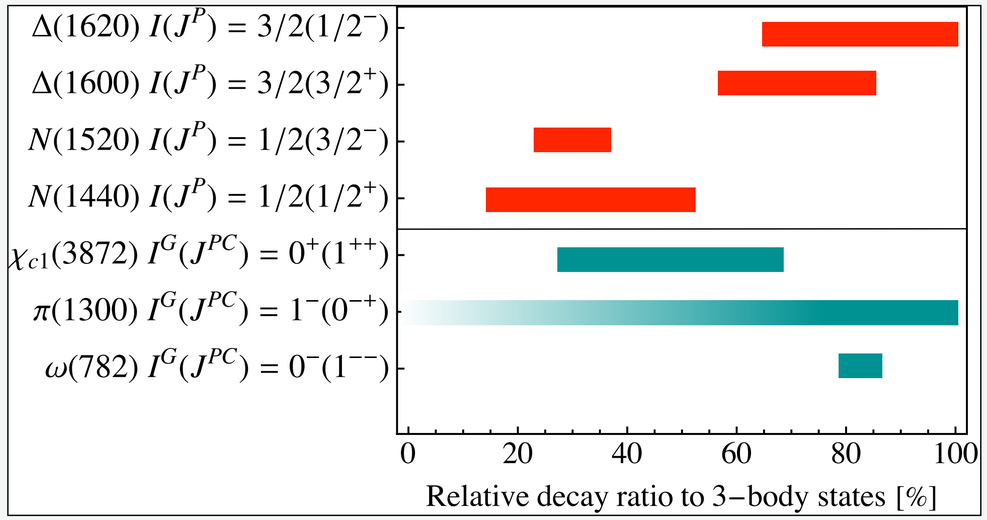
# BOUNDARY
# OF UNDERSTANDING

# HADRON SPECTRUM




- 70y research: $\Delta(1232), \rho(770), \omega(782), \ldots$
- Ongoing progress, new techniques and experiments
- Many overlapping and mostly excited states

  ≈ 100 mesons + 50 baryons (****)




- QCD governs the emergence of hadron spectrum
- Non-perturbative regime
- Many states have dominant/large three-body content

Review: MM/Meißner/Urbach Phys.Rept. 1001 (2023) 1-6

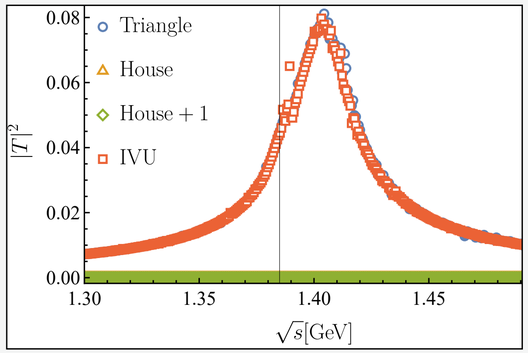
# 3-HADRON INTERACTIONS (IMPACT)

## Consistency frontier

- Why is excited pion so heavy? $\pi(1300)$ (later)
- Why is excited nucleon so light? $N(1440)$

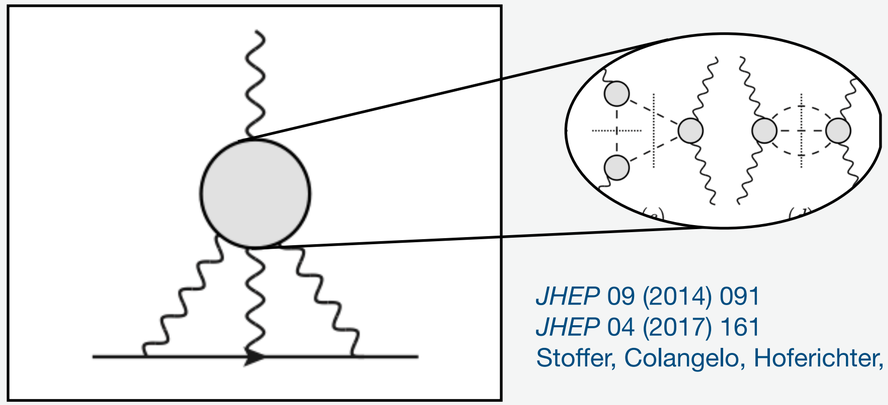
## Discovery frontier

- Spin-exotic states
- Left-hand-cuts: $T_{cc}(3875)$, etc.
- Triangle singularities: $a_1(1420)$, etc. (later)




Sakthivasan et al. *JHEP* 10 (2024) 246

GlueX@JLAB, COMPASS@CERN, …



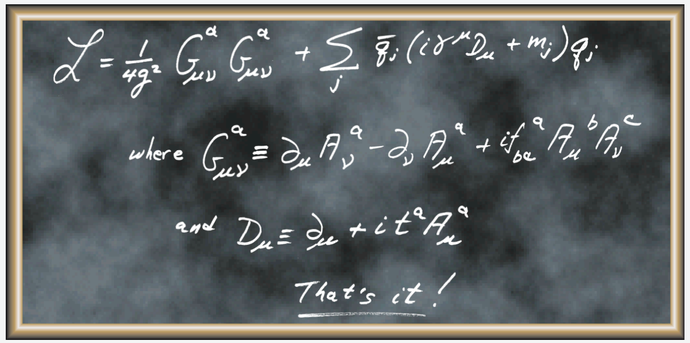
*JHEP* 09 (2014) 091
*JHEP* 04 (2017) 161
Stoffer, Colangelo, Hoferichter, …

## Precision frontier

- Hadronic light-by-light
- $\tau$-EDM
- …

Belle@SuperKEKB, …

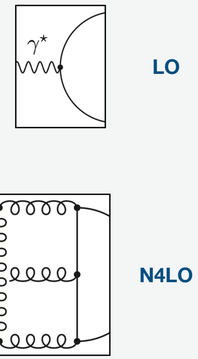
# THEORY




www.frankwilczek.com/Wilczek_Easy_Pieces/298_QCD_Made_Simple.pdf

## Quantum Chromodynamics (QCD)

- Degrees of freedom: quarks and gluons
- pQCD: well-defined set of rules (Feynman diagrams) to calculate transition rates

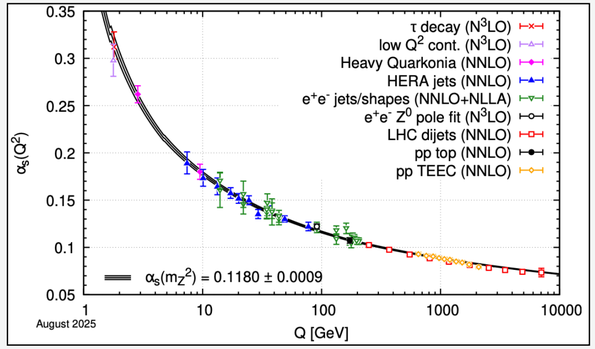



*JHEP* 08 (2017) 113

## Hadron spectroscopy

- Virtual effects: $\alpha_s$ depends on the momenta
  - $\alpha_s(Q) > 1$ *for small exchanged momenta*
  - *asymptotic states = stable hadrons* $p, n, \pi, \ldots$

←Spectroscopy pQCD→



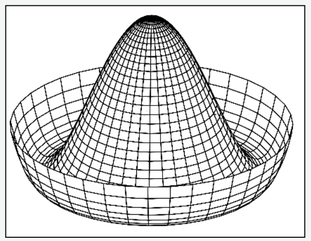

*Particle Data Group 2025*

# NON-PERTURBATIVE APPROACHES

## Effective Field Theory (EFT/CHPT)

Weinberg (1979) Gasser, Leutwyler (1981), …

Reviews: Ann. Rev. Nucl. Part. Sci. 57, 33 (2007), Adv. Nucl. Phys. 27, 277 (2003), …



$$Z[J] = \int [DU] e^{\int id^4x \; \mathscr{L}_{\text{eff}}(U,v,a,s,p)}$$

**[+] Effective/Hadronic degrees of freedom**

**[-] Infinitely many, unknown *low-energy constants***

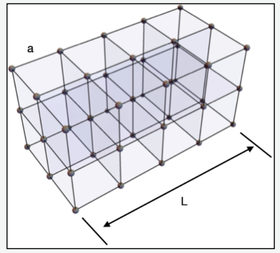
**[+] Well-defined power counting**

**[+] quark mass dependence**

## Lattice Gauge Theory (LQCD)

Wilson, Phys. Rev. D10 (1974) 2445 , …

Reviews: hep-lat/9807028 [hep-lat] Rev.Mod.Phys. 90 (2018) Rept.Prog.Phys. 86 (2023)




$$Z[J] = \int [DU] e^{-S_E} \det[M[U]]$$

**[+] QCD degrees of freedom**

**[-] discretized space-time / Euclidean metric / finite-volume**

**[+/-] unphysical quark masses**

**[+] non-perturbative access to QCD Green's functions**

## → S-matrix theory ←

*Crossing symmetry — particle/antiparticle*

*Unitarity — probability conservation*

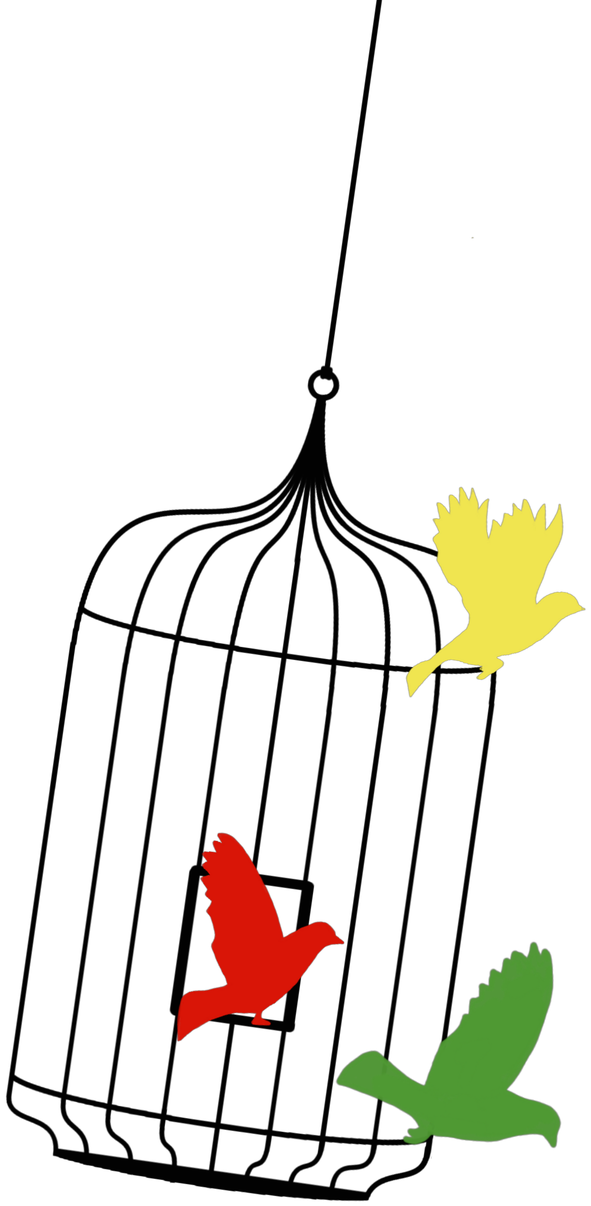
*Analyticity — causality*

**Further approaches: Functional methods, holography, K-matrix, dynamical models, …**

Review: Eichmann/Sanchis-Alepuz/Alkofer/Fischer Prog.Part.Nucl.Phys. 91 (2016) 1-100

Review: MM/Meißner/Urbach Phys.Rept. 1001 (2023) 1-6

Review: Döring/Haidenbauer/Sato/MM PPNP(2025)



# “EXPERIMENT”

# IN

# A BOX

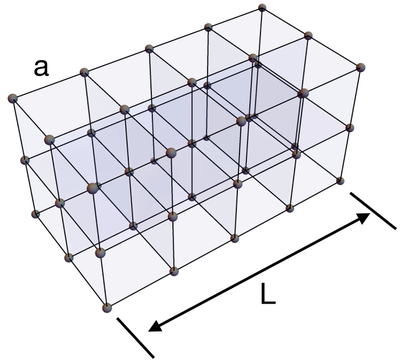
# LATTICE QCD (SPECTROSCOPY)

## Advantages

- QCD degrees of freedom
- Experimentally inaccessible scenarios:
  - → Unconventional quantum numbers
  - → Three-body scattering/… (ex. later)
  - → Chiral trajectory (ex. later)

## Roadblocks

- discretized (Euclidean) space-time — continuum extr.
- unphysical quark mass — extrapolations via EFTs
- **finite volume — quantization conditions needed**




Discretization of space-time

Euclidean space-time

Boundary conditions

Gauge and fermion degrees of freedom

plaquettes

Nielsen–Ninomiya theorem

Fermion doublers

links

Lattice QCD action

Construction of the action

$$Z[J] = \int [DU] e^{-S_E} \det[M[U]]$$

Wick's theorem

measure of integration in the path integral.

Generating functional

Hybrid Monte-Carlo simulation

The transcription of the operators used to probe the physics

Operator construction

Correlation functions

Generalized eigenvalue problem

scale setting

Energy eigenvalues ($\mathbb{R}$)

K. Wilson, Phys. Rev. D10 (1974) 2445 , …
Introduction to lattice QCD: Course Rajan Gupta hep-lat/9807028 [hep-lat] ….

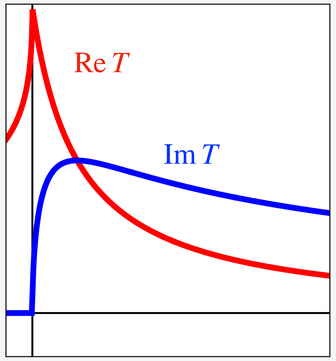
# QUANTIZATION CONDITION: 2-BODY EXAMPLE

Review: MM/Doring/Rusetsky Eur.Phys.J.ST 230 (2021)

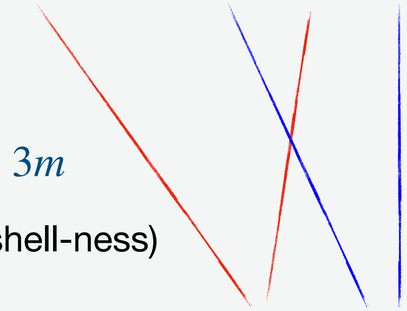
## Continuum QFT

- Asymptotic states at $t \to \pm\infty$
- underlying quantity: $(\sigma \sim |S|^2, \dots)$

$$S = 1 + iT \in \mathbb{C}$$







$$\text{disc}\, T \sim |T|^2 \quad \sqrt{s} < 3m$$

unitarity condition (on-shell-ness)

$$T^{-1} = K^{-1} - \int_l \frac{1}{2E_l} \frac{1}{(s - 4E_l^2 + i\epsilon)} = p \cot\delta - \left( \int \dots - Re \int \dots \right)$$

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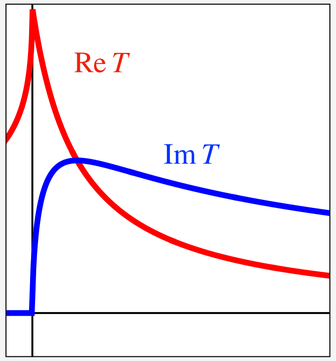



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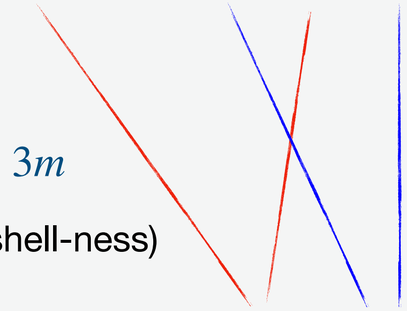

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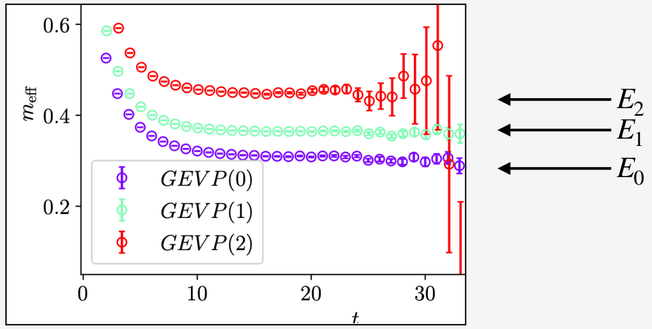
## Finite-volume setup

- Input: energy eigenvalues $\{E_0, \dots\} \in \mathbb{R}$
- Correlation functions at $it \to \infty$




- off-shell configurations decay exponentially $\sim O(e^{-ML})$
- on-shell states propagate/feel box-size $\sim O((ML)^n)$

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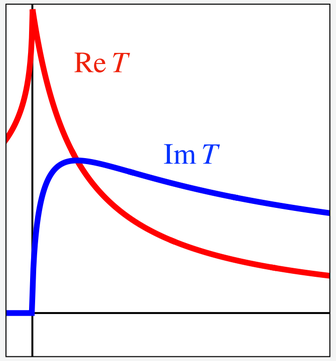



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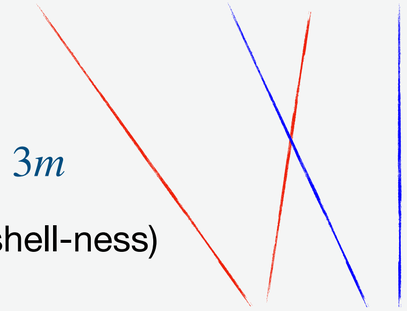

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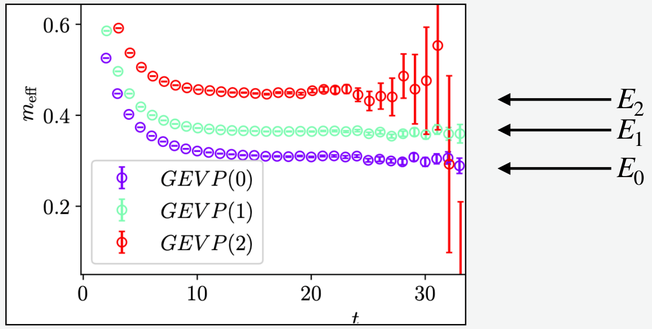


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$$T_{FV}^{-1} = p\cot\delta - \left(\frac{1}{L^3}\sum_{\vec{l}\in 2\pi/L\mathbb{Z}} \dots - Re\int_{\vec{l}} \dots\right) = 0$$

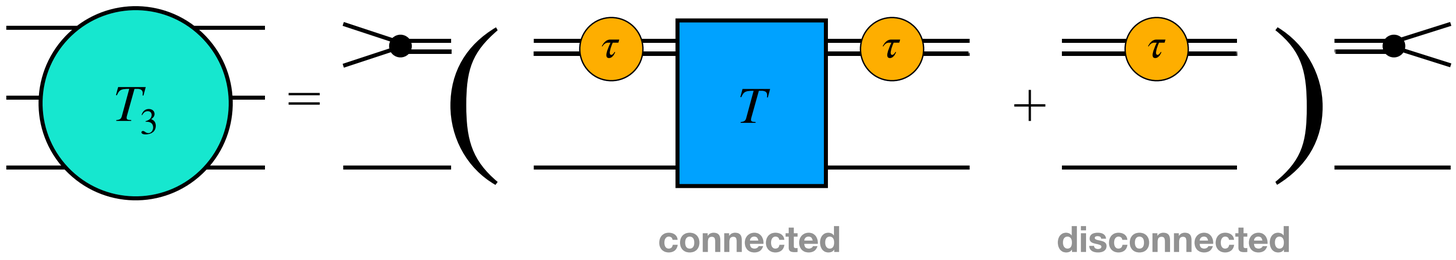
$$p\cot\delta(E_i) = Z_{00}(E_i) + \mathcal{O}(e^{-ML})$$

M. Lüscher, Nucl. Phys. B 354, 531 (1991)

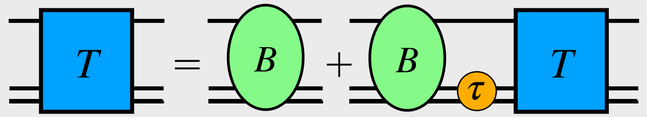
# 3-BODY INTERACTIONS

… same logic: start with a general scattering amplitude

MM/Hu/Döring/Pilloni/Szczepaniak Eur.Phys.J.A 53 (2017)



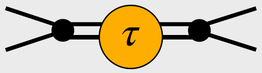

**Bethe-Salpeter ansatz**




… 4D integral equation

… unknown building blocks

**isobar**, aka **dimer**, aka **2-body subsystem**




- a tower of states for $L = 0,1,2,..$
- can be repulsive (!)

MM/Doring Phys.Rev.Lett. 122 (2019)

… imposing 3-body unitarity

Olive/Eden/… “The Analytic S-matrix”
Bedaque, Griesshammer (1999)
R. Aaron, R. D. Amado, and J. E. Young, Phys.Rev. 174, 2022 (1968)

# 3-BODY UNITARITY

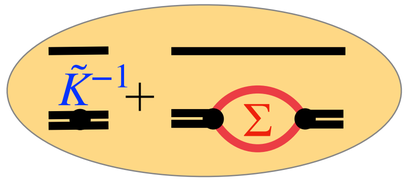
… imposing 3-body unitarity

$$\langle\{q\}\,|\,(\hat{T}_3-\hat{T}_3^\dagger)\,|\,\{p\}\rangle = i\int\prod_{\ell=1}^{3}\frac{\mathrm{d}^4k_\ell}{(2\pi)^4}\,(2\pi)\delta^+(k_\ell^2-m^2)\,(2\pi)^4\delta^4\left(P-\sum_{\ell=1}^{3}k_\ell\right)\langle\{q\}\,|\,\hat{T}_3^\dagger\,|\,\{k\}\rangle\,\langle\{k\}\,|\,\hat{T}_3\,|\,\{p\}\rangle$$

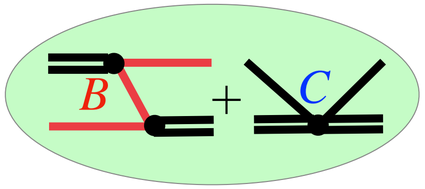
… exact solution — constraints on $\mathrm{disc}\,B$ and $\mathrm{disc}\,\tau^{-1}$

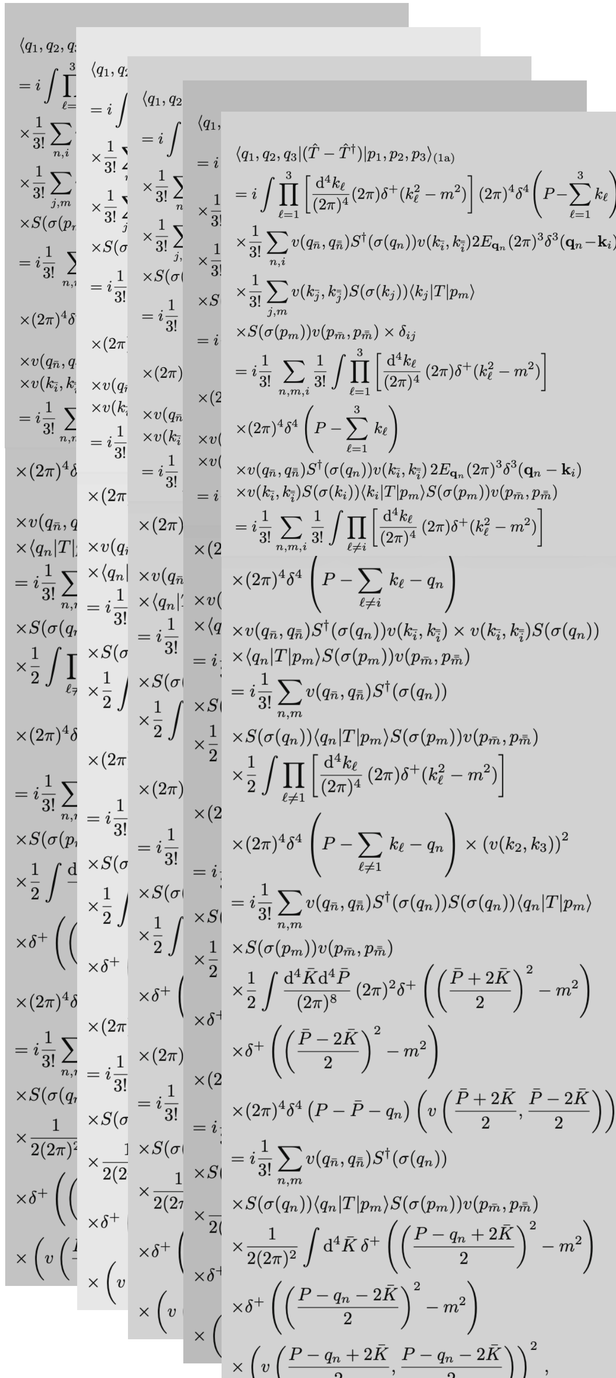
MM/Hu/Döring/Pilloni/Szczepaniak Eur.Phys.J.A 53 (2017)

$$\tau^{-1}(\sigma(k)) = A + B\,\sigma(k) + \frac{\sigma(k)^2}{\pi}\int_{4m^2}^{\infty} d\sigma'\,\frac{\mathrm{Im}\,\tau^{-1}(\sigma')}{\sigma'^2(\sigma'-\sigma(k)-i\epsilon)}$$




$$B(s,p,p') = -\frac{v(Q,q)v(Q,p)}{2\sqrt{m^2+\mathbf{Q}^2}\left(E_Q-\sqrt{m^2+\mathbf{Q}^2}+i\epsilon\right)}$$



## Three-body scattering amplitude

- 3b-Unitarity
- intergal equation (explicit momenta)
- generalized to all channels + strangeness

Feng+ *PRD* 110 (2024) 094002 2407.08721

### "Infinite Volume Unitarity" (IVU)

$$T(s,p,p') = B + C + \int \frac{d^3\ell}{(2\pi)^3} \frac{(B+C)}{2E_l} \frac{1}{\tilde{K}^{-1} - \Sigma} T$$

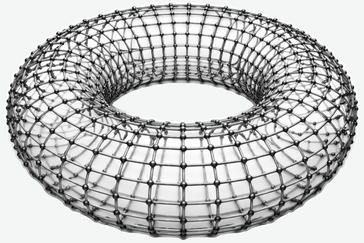
MM/Hu/Döring/Pilloni/Szczepaniak Eur.Phys.J.A 53 (2017)


**on-shell** configurations
no free parameters


**Real/offshell**
"2-/3-body force"

## Three-body quantization condition

- only discrete momenta are possible $2\pi/L\,\mathbb{Z}^3$
- determinant equation
- relevant volume-dependence: singularities/onshell states $(+O(e^{-ML}))$



$\sim O((ML)^n)$ $\sim O(e^{-ML})$

### "Finite Volume Unitarity" (FVU)

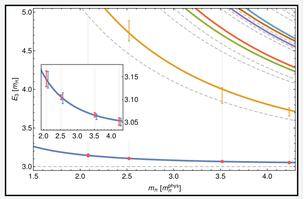
$$\det\left[\, 2L^3 E_{\mathbf{p}} \left(\tilde{K}^{-1} - \Sigma^L\right) - B - C \,\right]^{\Gamma} \equiv 0$$

MM/Döring Eur.Phys.J.A 53 (2017) 12, 240

Related: Poleyaeva (2012), Hansen+ (2014), Hammer+ JHEP 10 (2017) 115, Wunderlich+ JHEP 08 (2019); Jackura+ Eur.Phys.J.C 79 (2019); Kang Yu+ 2603.07205 [hep-lat], Dawid+ Phys.Lett.B 864 (2025) 139442… Reviews: Hansen/Sharpe: Ann.Rev.Nucl.Part.Sci. 69 (2019) 65-107, MM/Rusetsky/Doring *Eur.Phys.J.ST* 230 (2021) 6, 1623-1643

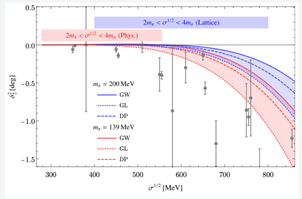
$\pi^+\pi^+\pi^+$

Phys.Rev.Lett. 122 (2019)



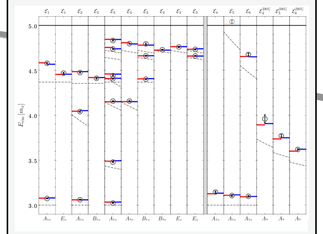
$\pi^+\pi^+\pi^+$

*Phys.Rev.D* 101 (2020) 5



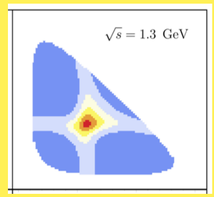
$\pi^+\pi^+\pi^+$

*Phys.Rev.D* 101 (2020) 11



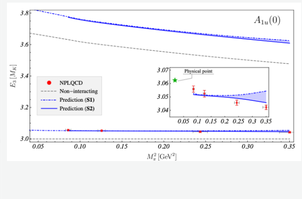
$a_1(1260)$

*Phys.Rev.D* 101 (2020) 9



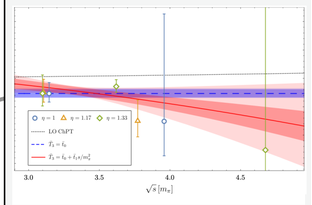

$K^-K^-K^-$

*Phys.Rev.D* 102 (2020) 11



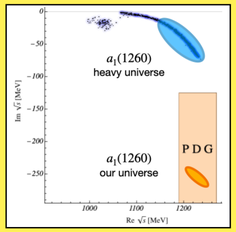
$\pi^+\pi^+\pi^+$

*Phys.Rev.D* 104 (2021)



$a_1(1260)$

Phys.Rev.Lett. 127 (2021)

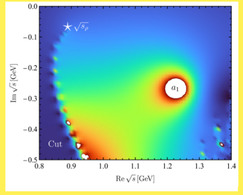



Eur.Phys.J.A 53 (2017) 12, 240

# FVU/IVU APPLICATIONS

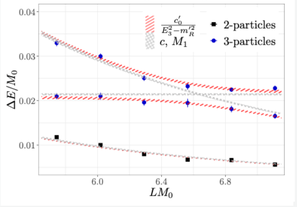
$a_1(1260)$

*Phys.Rev.D* 105 (2022)



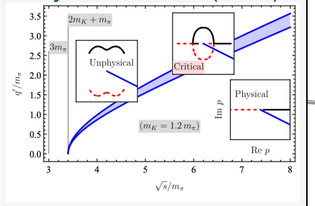
$3\phi$

*JHEP* 02 (2023) 252



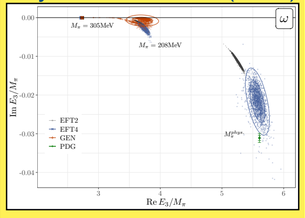
$3 \times (\pi/K)$

*Phys.Rev.D* 110 (2024)



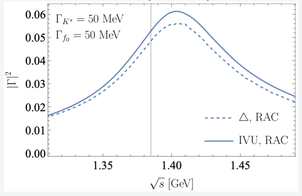

$\omega(782)$

Phys.Rev.Lett. 133 (2024)



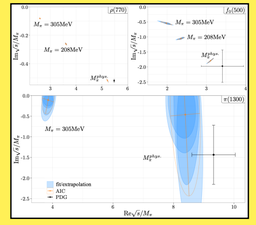
Triangle sing.

*JHEP* 10 (2024) 246



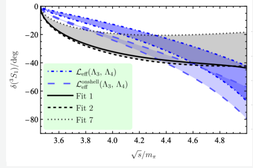
$\pi(1300)$

2510.09476 in press PRL



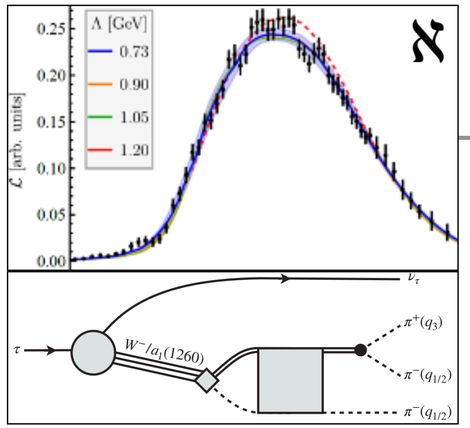
Isotensor $\pi\pi\pi$

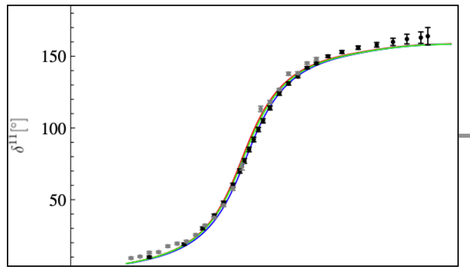
2601.16916



# AXIAL-VECTOR MESON

Sadasivan/Akdag/MM/Culver/Alexandru/Lee/Doring Phys.Rev.D 101 (2020); *Phys.Rev.D* 105 (2022)



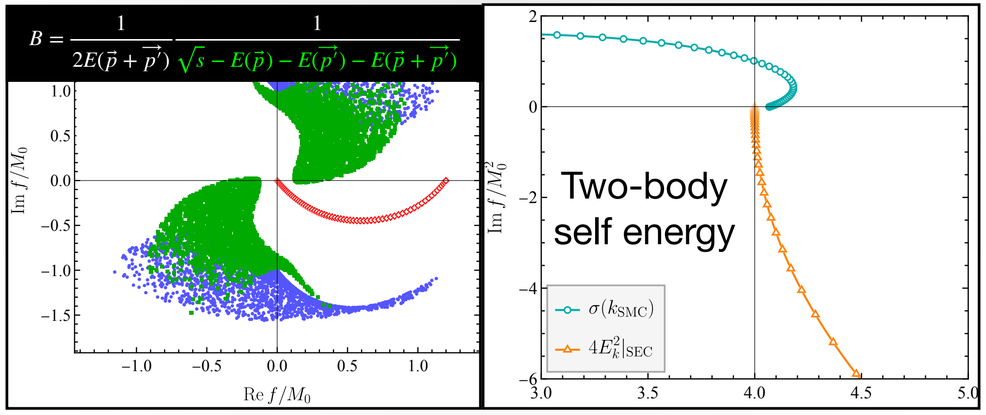

Schael et al. [ALEPH] Phys.Rept. 421 (2005); Nucl.Phys.B 79; Phys.Rev.D 7;

**IVU**

$$T(s,p,p') = B + C + \int \frac{d^3\ell}{(2\pi)^3} \frac{(B+C)}{2E_l} \frac{1}{\tilde{K}^{-1} - \Sigma} T$$

## Solution technique

- Complex contour deformation $|\ell| \in \mathbb{C}$
- Analytic continuation to the real axis




Hetherington/Schick, Phys. Rev. 137, B935–B948 (1965). Cahill/Sloan, Nucl. Phys. A 165, 161–179 (1971) Schmid/Ziegelmann, Adhikari/Amado, Phys. Rev. D 9, 1467–1475 (1974)

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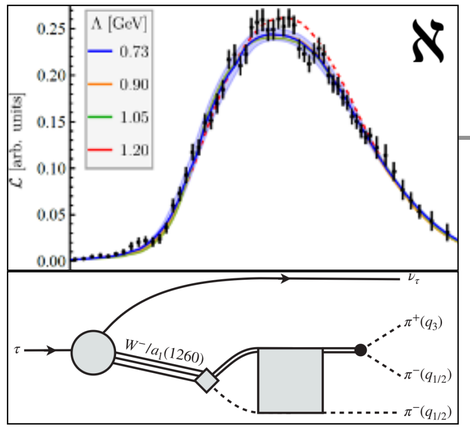


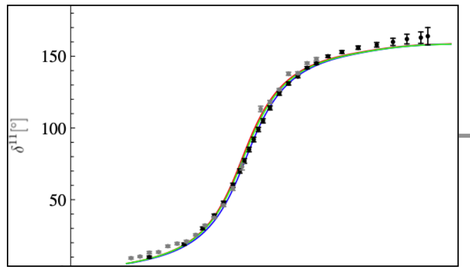


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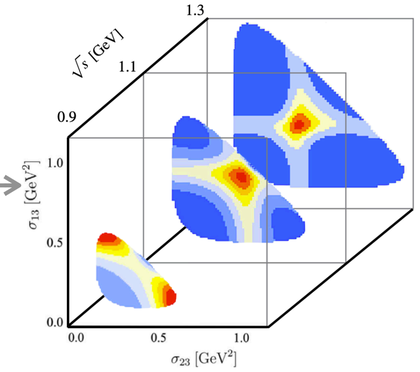


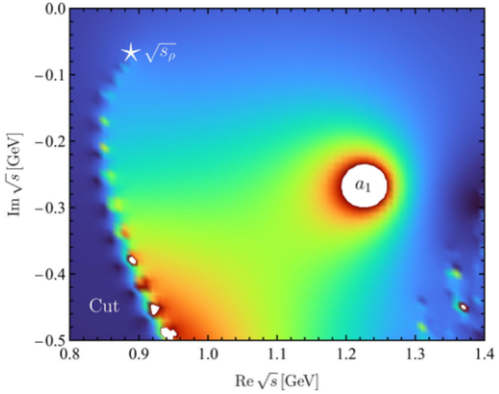


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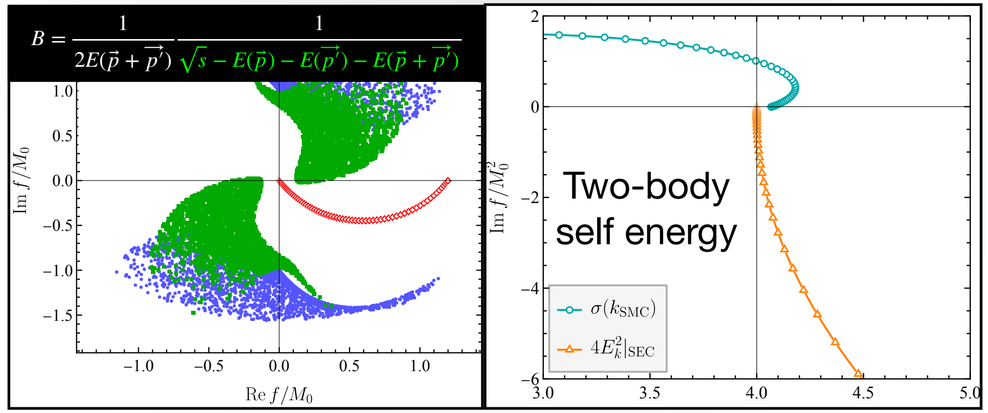


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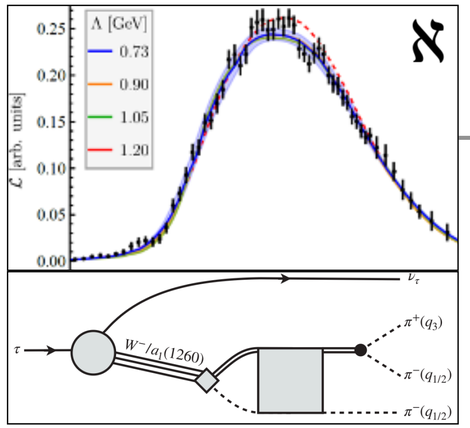


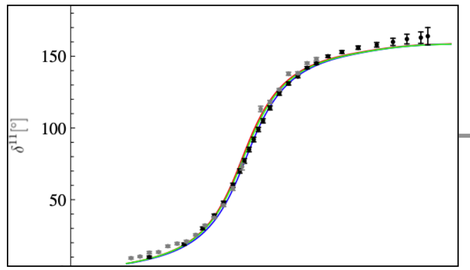


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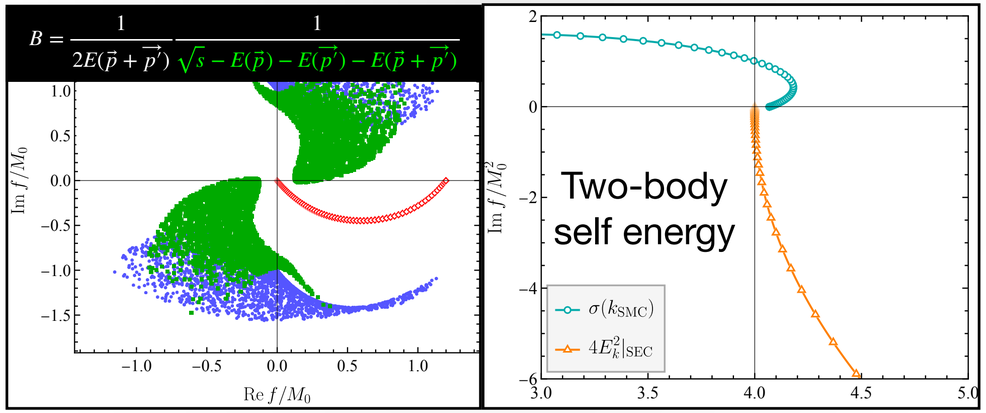


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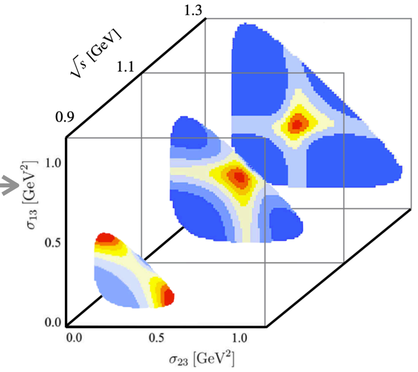
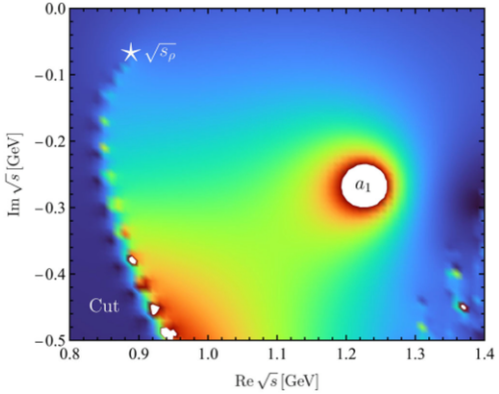
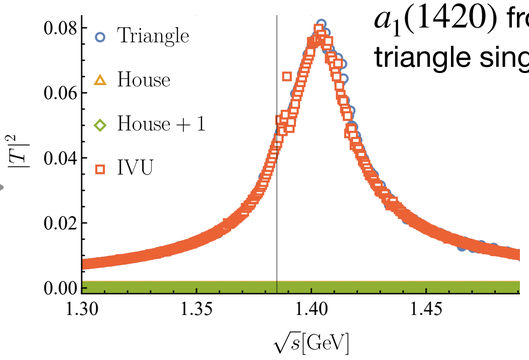



$a_1(1420)$ from triangle singularity

Sakthivasan et al. *JHEP* 10 (2024) 246

# AXIAL-VECTOR MESON (LATTICE QCD)

MM/Culver/Alexandru/Doring/Lee/Sadasivan [GWQCD] PRL 127 (2021)

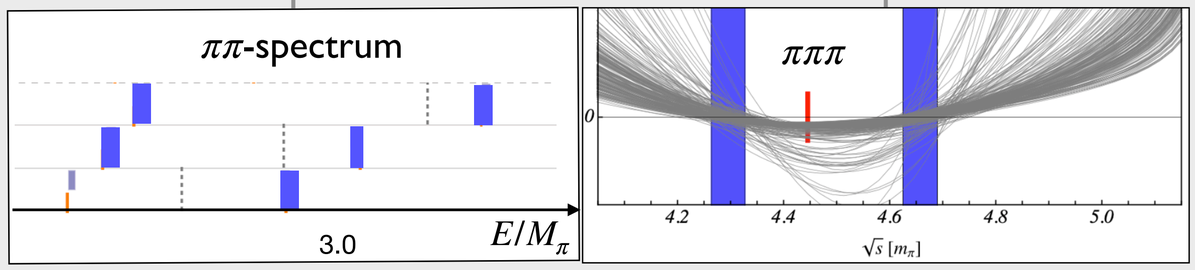
**IVU**

$$T(s,p,p') = B + C + \int \frac{d^3\ell}{(2\pi)^3} \frac{(B+C)}{2E_l} \frac{1}{\tilde{K}^{-1} - \Sigma} T$$

$C, \tilde{K}$
volume independent
generic form

**FVU**

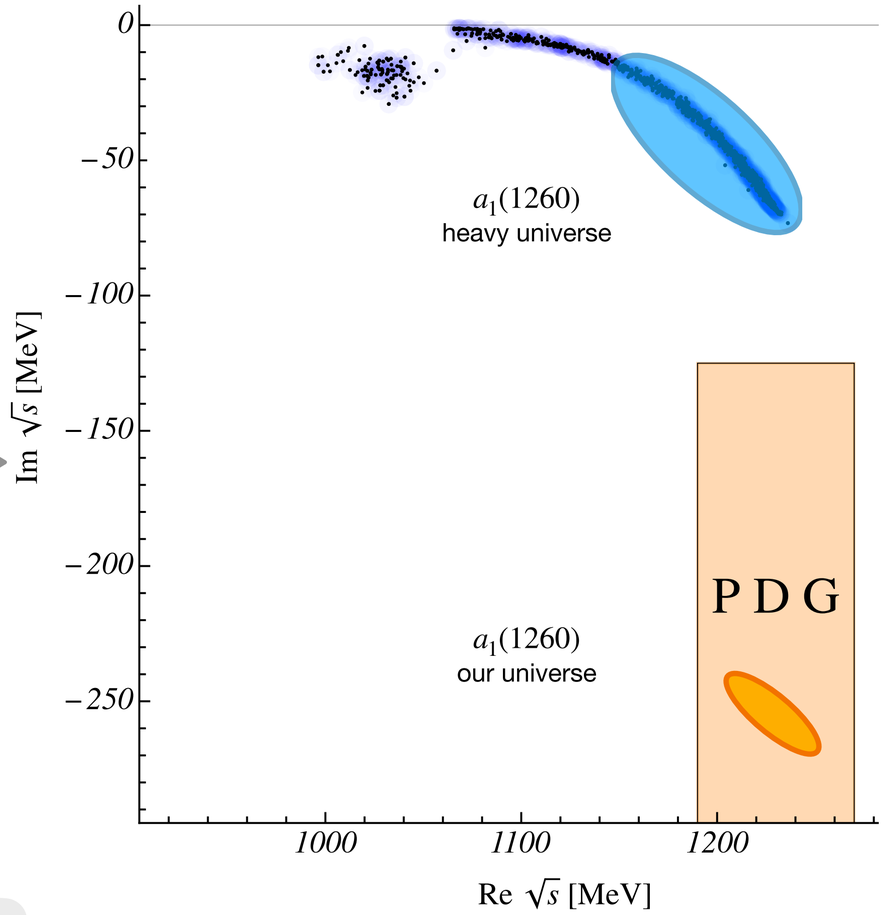
$$\det\left[\, 2L^3 E_{\mathbf{p}} \left(\tilde{K}^{-1} - \Sigma^L\right) - B - C \,\right]^{\Gamma} \equiv 0$$




## Lattice QCD

- $M_\pi \approx 315\text{MeV}$
- $N_f = 2$

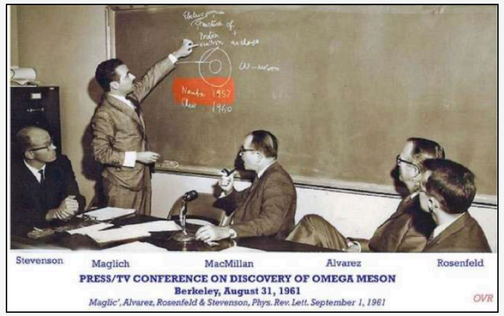
[GWQCD] PRD94(2016) PRD98 (2018) PRD 100(2019)



# VECTOR MESON

$\omega(782)$

- lightest hadron decaying into three particles




- isoscalar response within the VMD of the photon-nucleon interactions
- ~repulsion at < 1 fm in the one-boson-exchange picture of the N–N interaction

Sakurai (1960); Erkelenz (1974); Brown and Jackson (1976); Barkov et al., 1985; Connell et al. (1997); Bazavov et al. (2021)

**What can we learn from Lattice QCD?**

- two/three-body force
- pion-mass dependence
- KSFR/Universality relations/… in EFT

Gell-Mann/Sharp/Wagner/Fujiwara/Kawarabayashi… Review: Meißner Phys.Rept. 161 (1988) 213

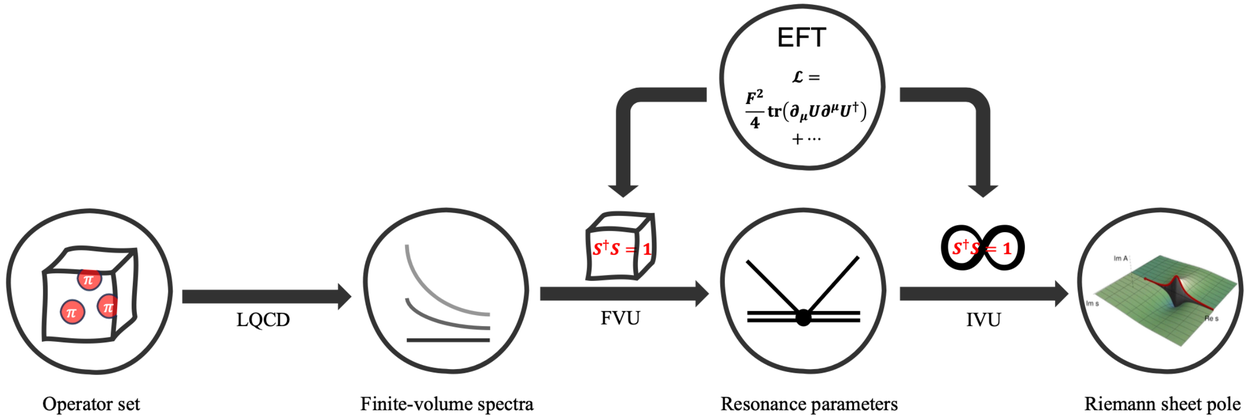
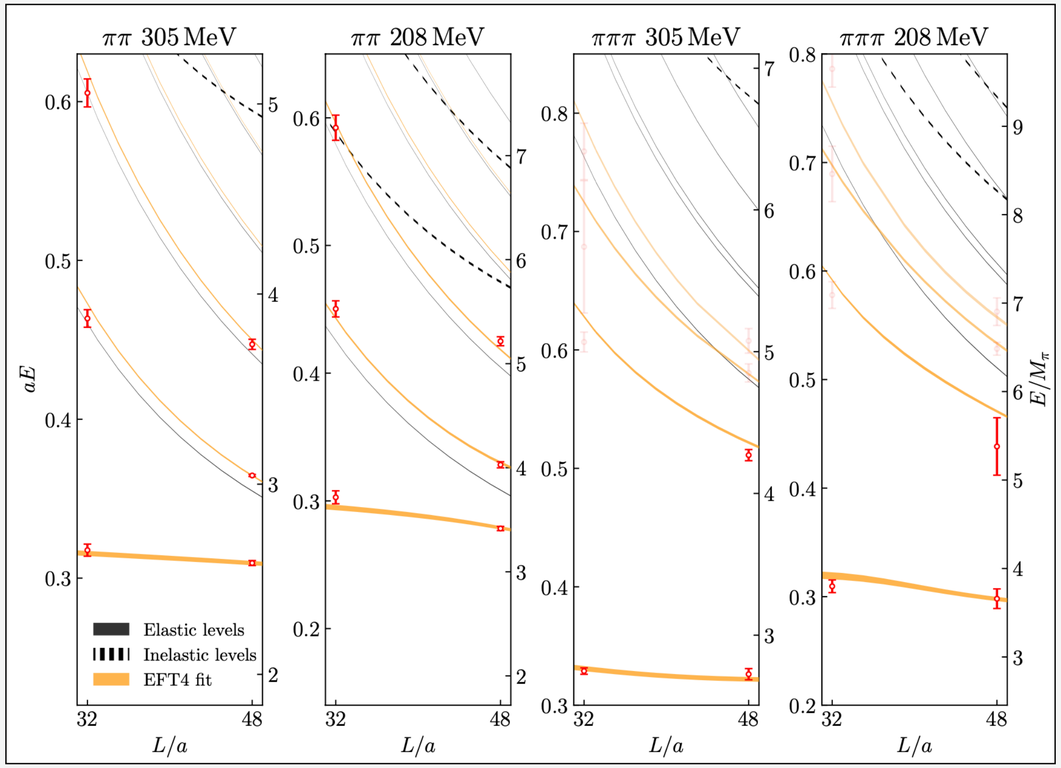



Fig by Haobo Yan (燕浩波)

# VECTOR MESON

**Lattice QCD input**

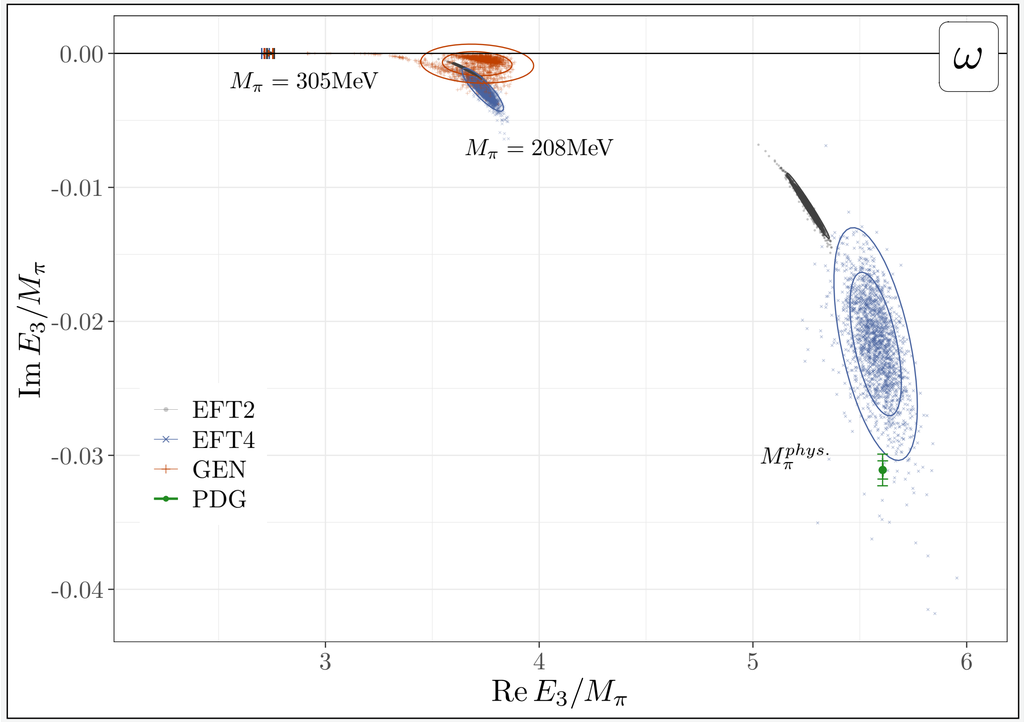
- Nf = 2 + 1 Clover fermions
- 2/3 particle operators
- 2 pion masses ( $\approx 210, 305$ MeV) 2 volumes ($L^3 = 32^3, 48^3$)




**FVU**

$$\det\left[\, 2L^3 E_{\mathbf{p}} \left(\tilde{K}^{-1} - \Sigma^L\right) - B - C \,\right]^{\Gamma} \equiv 0$$

- Various EFT based ansatzes
- $\omega(782)$ becomes abound state at ~300 MeV
- at the physical point very close to the EXP value



# EXCITED PION

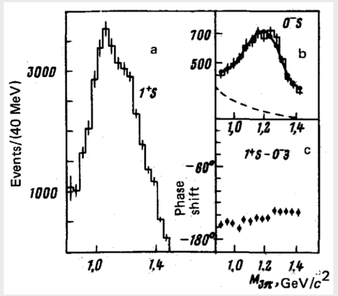
## $\pi(1300)$ $I^G(J^{PC}) = 1^-(0^{-+})$

| | | |
|---|---|---|
| $\pi(1300)$ MASS | [1] | $1300 \pm 100$ MeV |
| $\pi(1300)$ WIDTH | [1] | 200 to 600 MeV |

$\pi(1300)$ **DECAY MODES**

| Mode | | Fraction ($\Gamma_i / \Gamma$) | Scale Factor/ Conf. Level | P(MeV/c) |
|---|---|---|---|---|
| $\Gamma_1$ | $\rho\pi$ | seen | | 404 |
| $\Gamma_2$ | $\pi(\pi\pi)_{S-\text{wave}}$ | seen | | |

**Discovery:** $\pi Z \to \pi\pi\pi Z$




Bellini+ [Bologna-Dubna-Milan-Collaboration] JETP Lett. 34 (1981) 488

**Most recent analysis** $D \to 4\pi$

- Strong correlations between PWA contributions

| | $m_{a_1(1260)}$ | $\Gamma_{a_1(1260)}$ | $m_{a_1(1640)}$ | $\Gamma_{a_1(1640)}$ | $m_{\pi(1300)}$ | $\Gamma_{\pi(1300)}$ |
|---|---|---|---|---|---|---|
| $m_{a_1(1260)}$ | $+1.000$ | $+0.689$ | $-0.065$ | $-0.282$ | $+0.116$ | $-0.258$ |
| $\Gamma_{a_1(1260)}$ | | $+1.000$ | $-0.114$ | $-0.176$ | $+0.013$ | $-0.004$ |
| $m_{a_1(1640)}$ | | | $+1.000$ | $-0.335$ | $-0.136$ | $-0.119$ |
| $\Gamma_{a_1(1640)}$ | | | | $+1.000$ | $-0.258$ | $+0.370$ |
| $m_{\pi(1300)}$ | | | | | $+1.000$ | $-0.425$ |
| $\Gamma_{\pi(1300)}$ | | | | | | $+1.000$ |

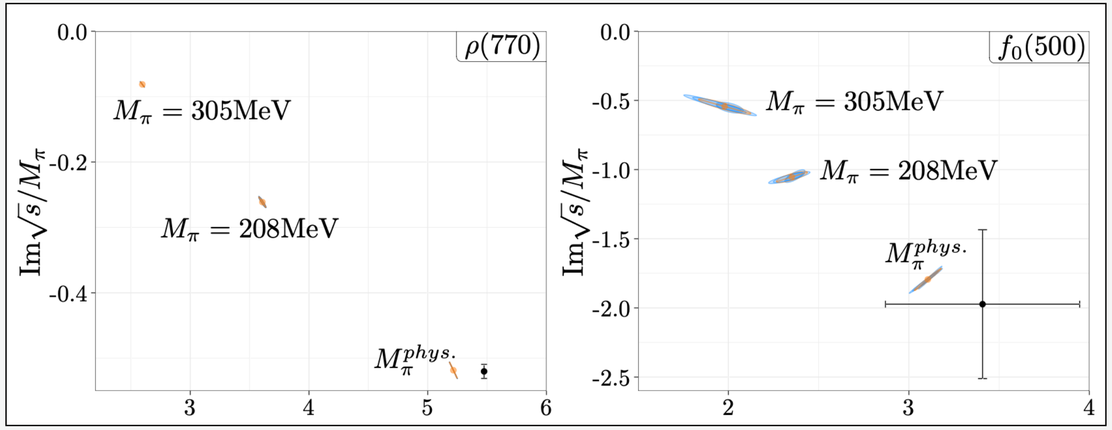
- Existence required by data?

d'Argent+ JHEP 05 (2017) 143

## Phenomenology

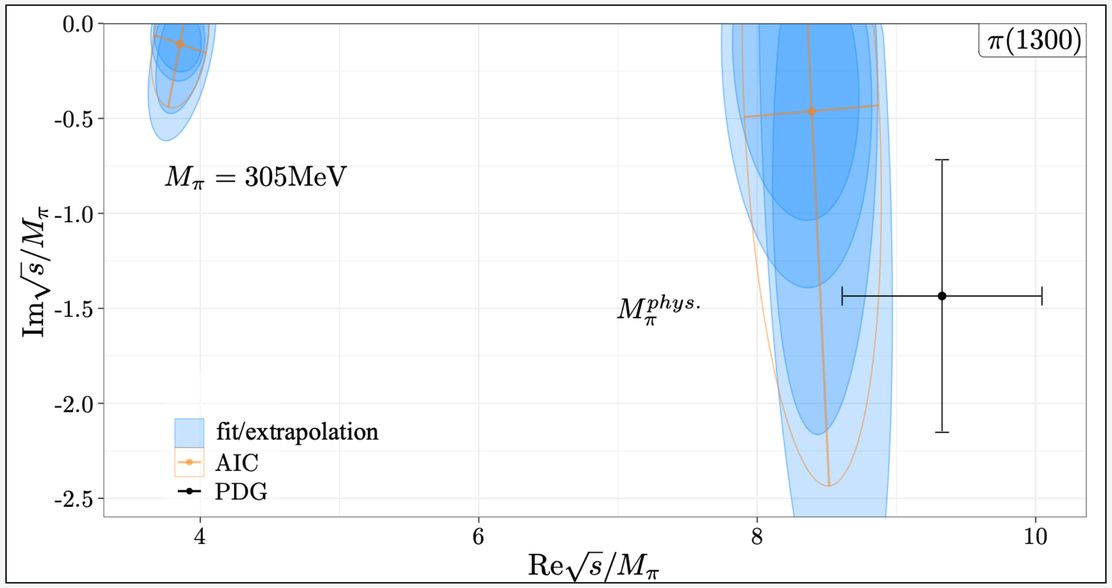
- Very heavy: $M_{\pi(1300)} \approx 10 M_\pi \approx M_{N(1440)}$
- Phenomenologically hard to access: overlapping effects $a_1(1260), \ldots$
- Width very uncertain

# $\pi(1300)$ POLE POSITIONS



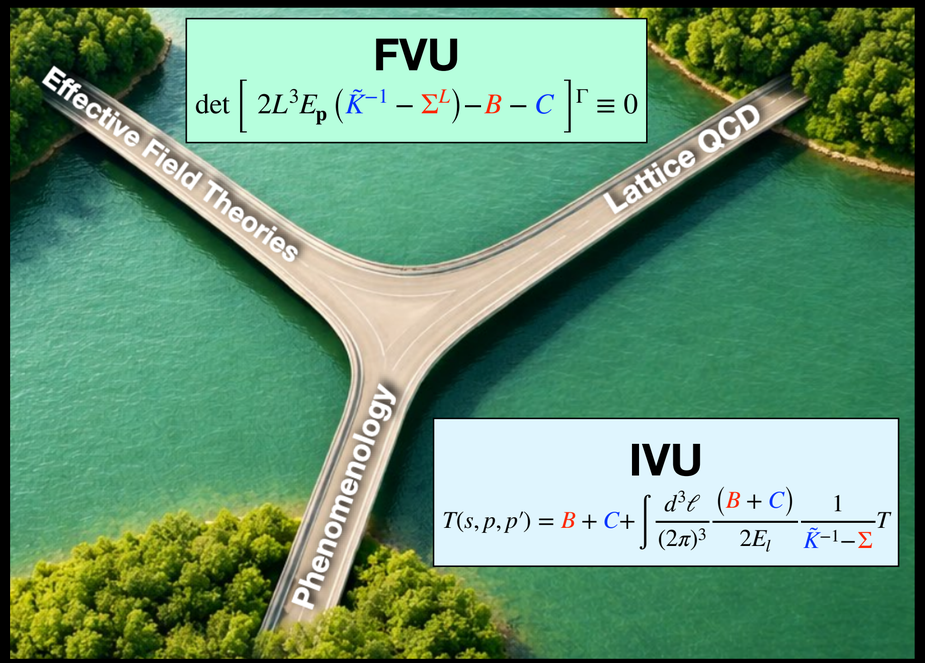

**2b resonances and chiral trajectories (in passing)**

- $f_0(500)$ agrees with PDG
- $\text{Re}\, M_{\rho(770)}$ slightly too low — Hoferichter+ PL B853 138698 (2024)
  - PDG value improved drastically
  - Discretization effects?
  - Over-constrained by mIAM?




**Excited state of the pion**

- $M_\pi = 305\,\text{MeV}$ finite-volume ensemble
  - ~~**no-resonance scenario**~~
  - Includes *dynamically generated* scenarios
- Physical point extrapolation
  - Mass/Width agree with PDG ($1\sigma$)

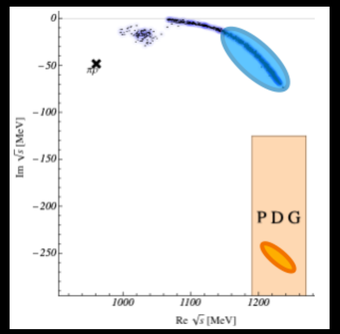


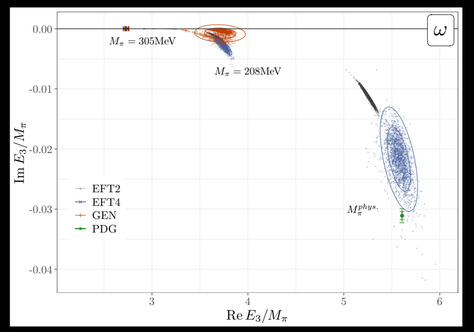

## New discoveries from QCD

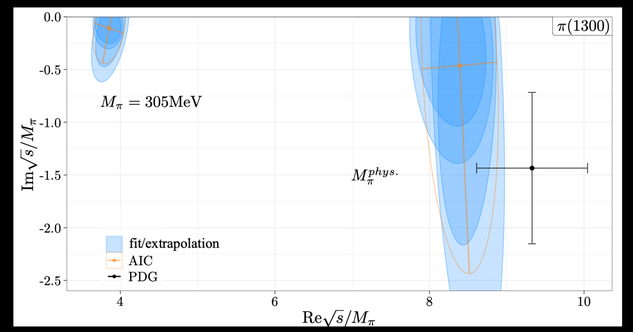
- Ab-initio Lattice input
- quark-mass extrapolations through EFT

## S-matrix & 3-body unitarity

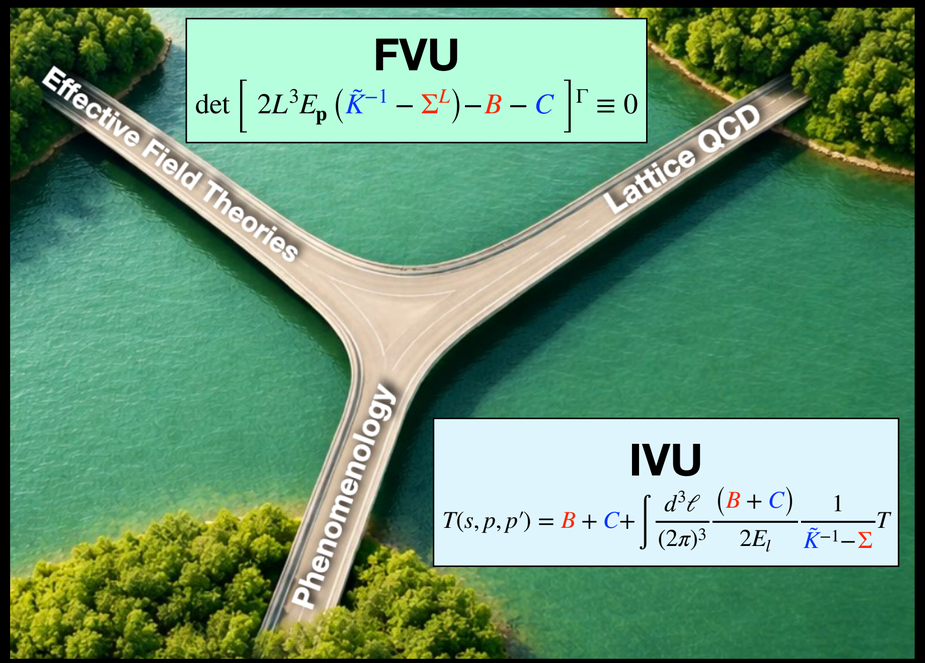
- On/off-shell configurations
- **new 3-body quantization condition**




# SUMMARY

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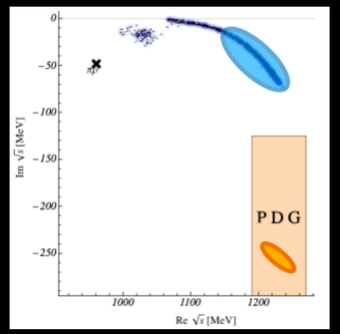


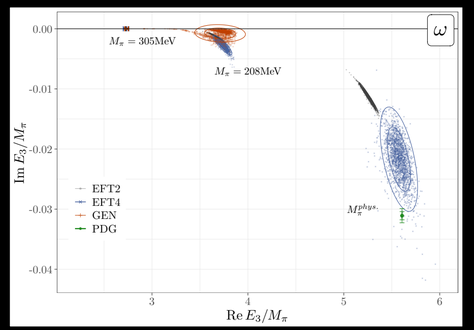


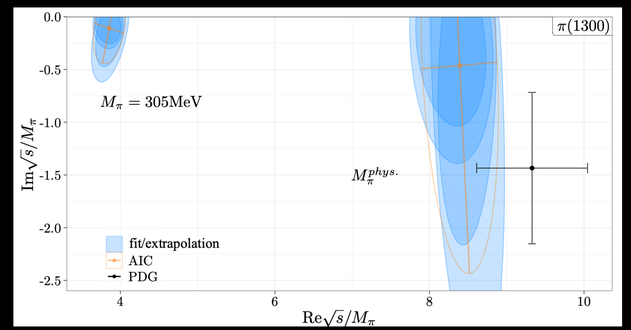


# SUMMARY

# OUTLOOK

*it is just the beginning!*

✳ Applications: $DD\pi$, $N(1440)$, . . . spin-exotics? — $a_1(1420)$

✳ FLAG ready: Systematics/statistics improvement, cutoff treatment…

✳ EFT tests: Universality of $\omega \to 3\pi, \rho \to 2\pi$ coupling, …

✳ **BEYOND SPECTROSCOPY (Feedback and ideas are welcome!)**