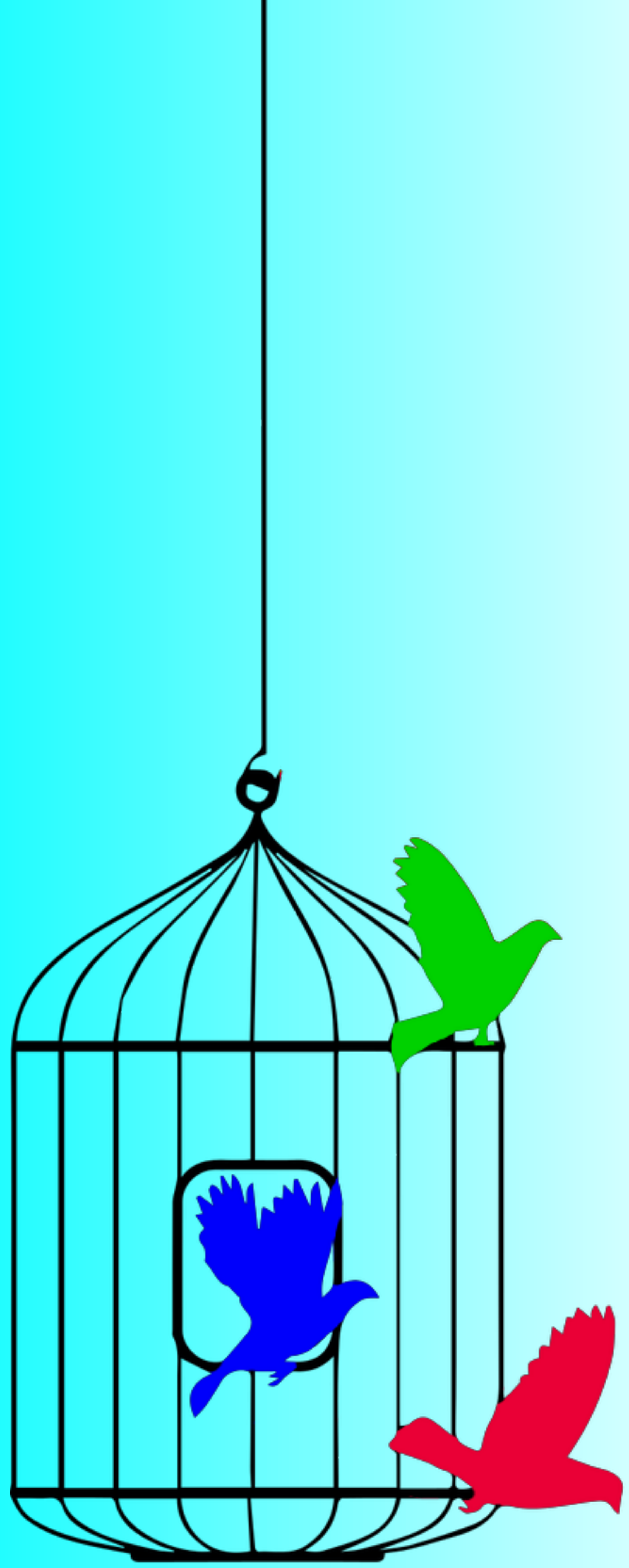


3-BODY PROBLEM FROM PHENOMENOLOGY AND LATTICE QCD

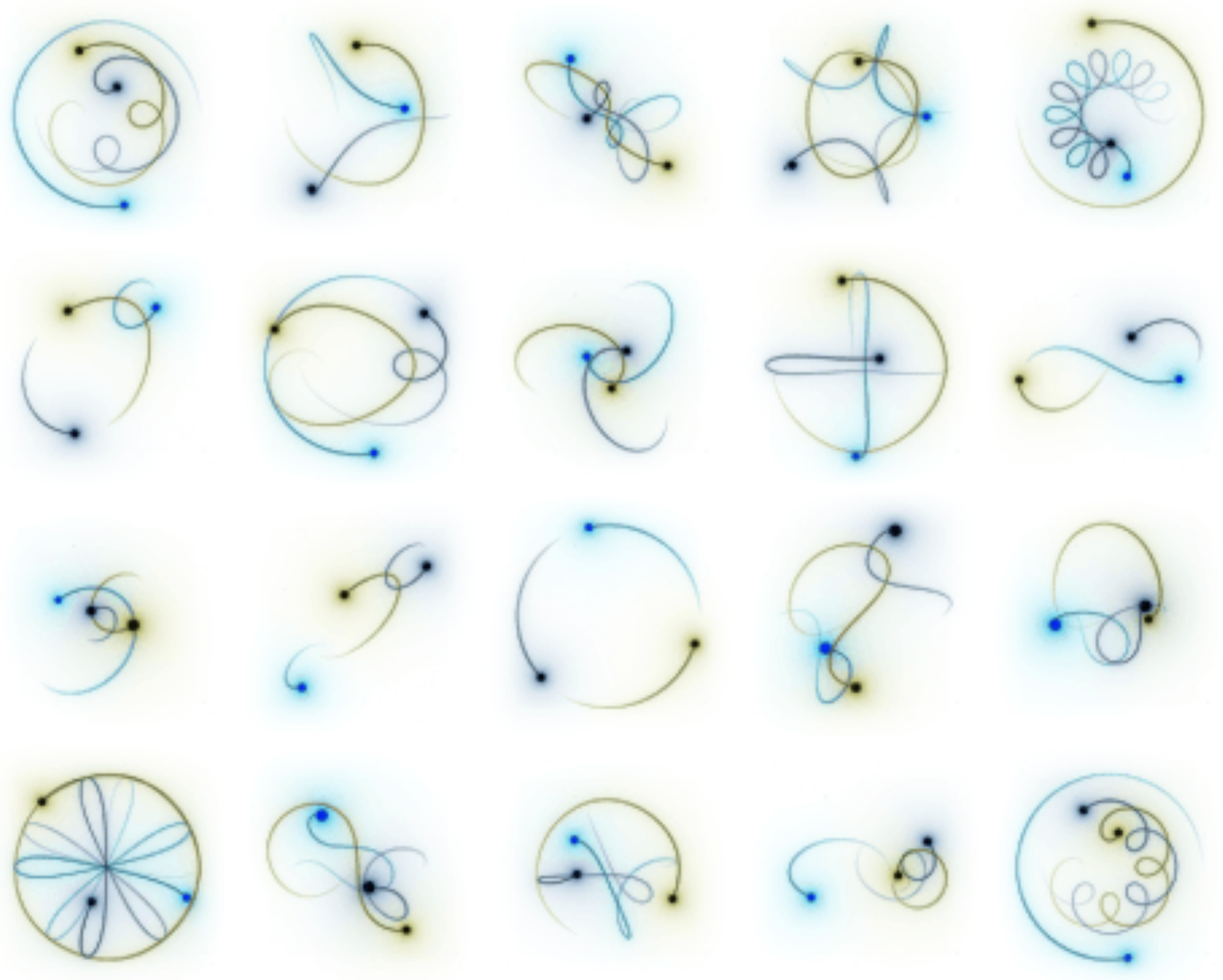
MAXIM MAI

University of Bonn | The George Washington University

Heisenberg-Programm
DFG Deutsche Forschungsgemeinschaft

NSF PHY-2012289
DOE DE-SC0016582
DOE DE-SC0016583
DFG CRC 110

THREE-BODY PROBLEM

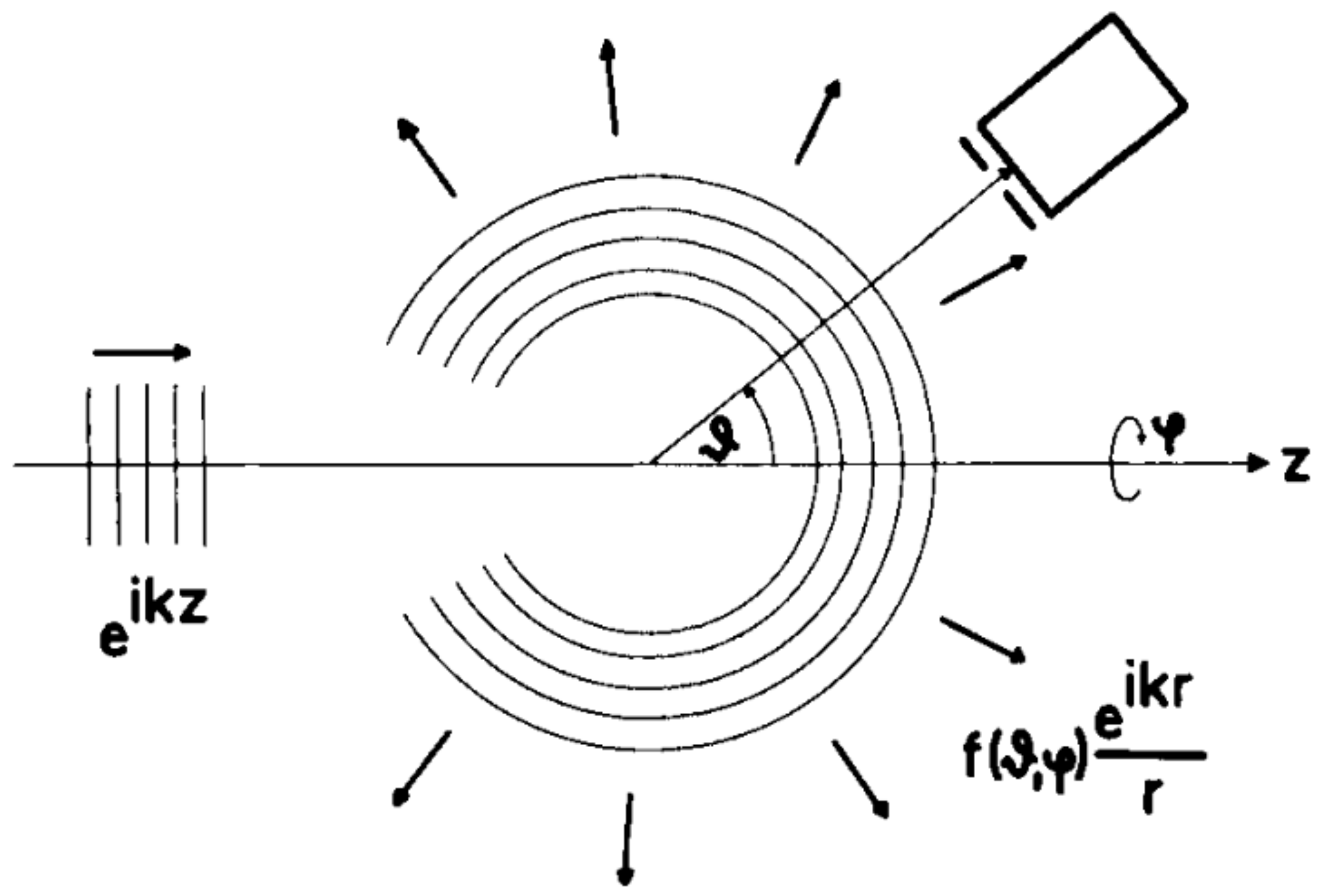


Gravitational three-body problem

- goal: space-time trajectories
- challenges:
  - no closed solutions
  - in general non-repeating (few exceptions[1])
- birth of mathematical chaos[2]

[1] Šuvakov/Dmitrašinović PhysRevLett.110.114301 (2013)
[2] Poincaré
[FIG] Adapted from Canagrisa, CC BY-SA 4.0 via Wikimedia Commons

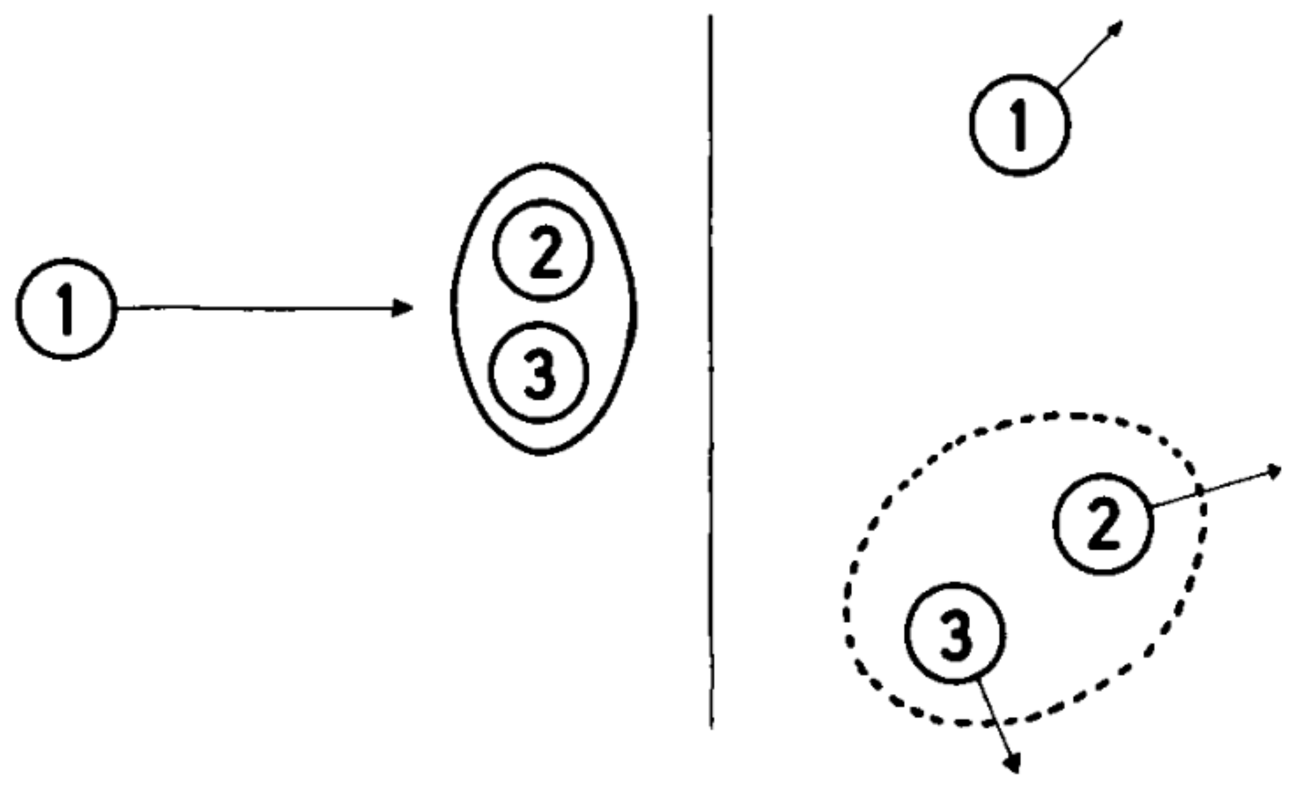
THREE-BODY PROBLEM



2 → 2

Quantum mechanical three-body problem

- goal: rigorous scattering theory
- challenges:
    - continuum of two-body scattering states[1]
    - 8 kinematic degrees of freedom



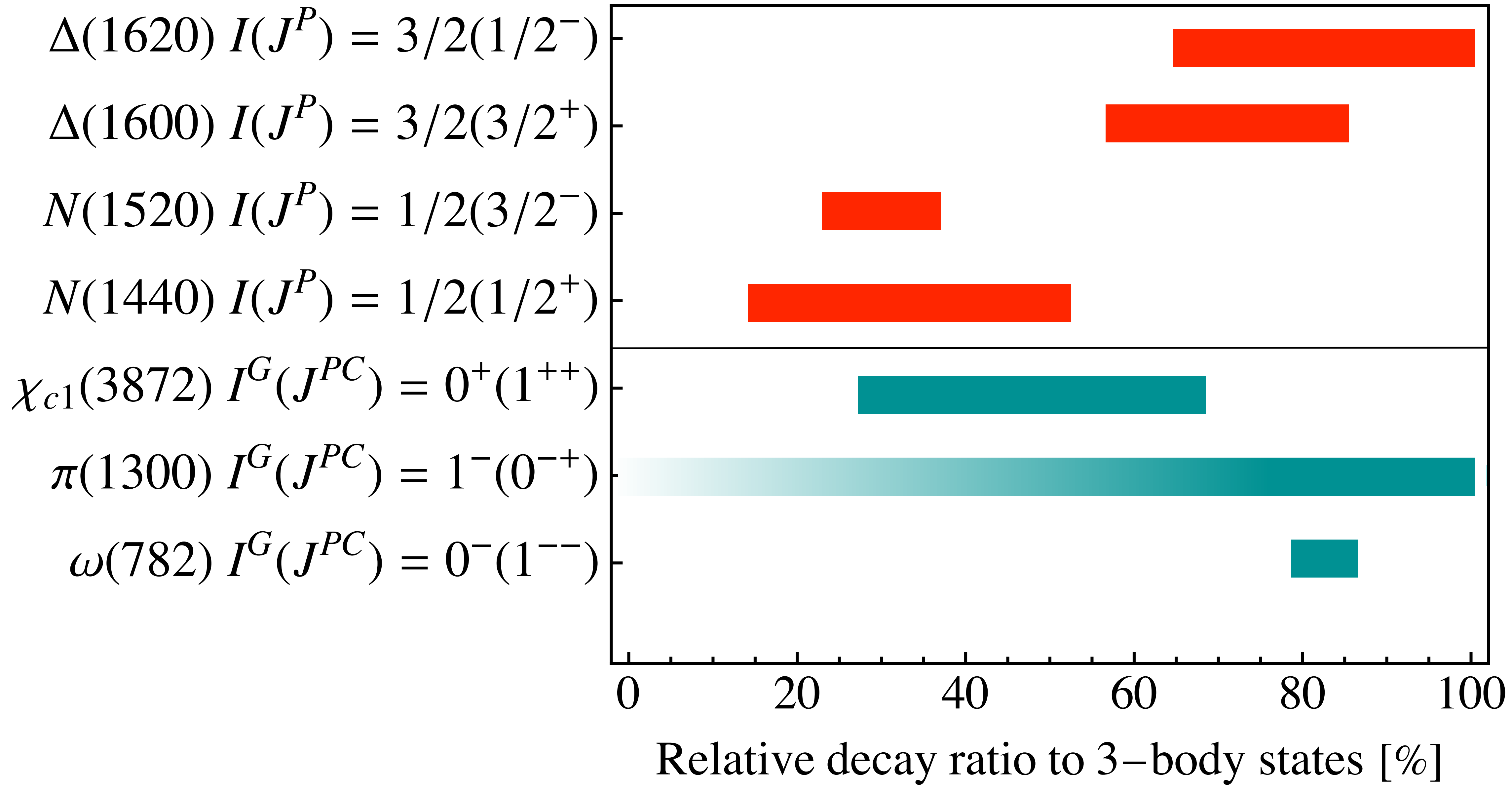
3 → 3

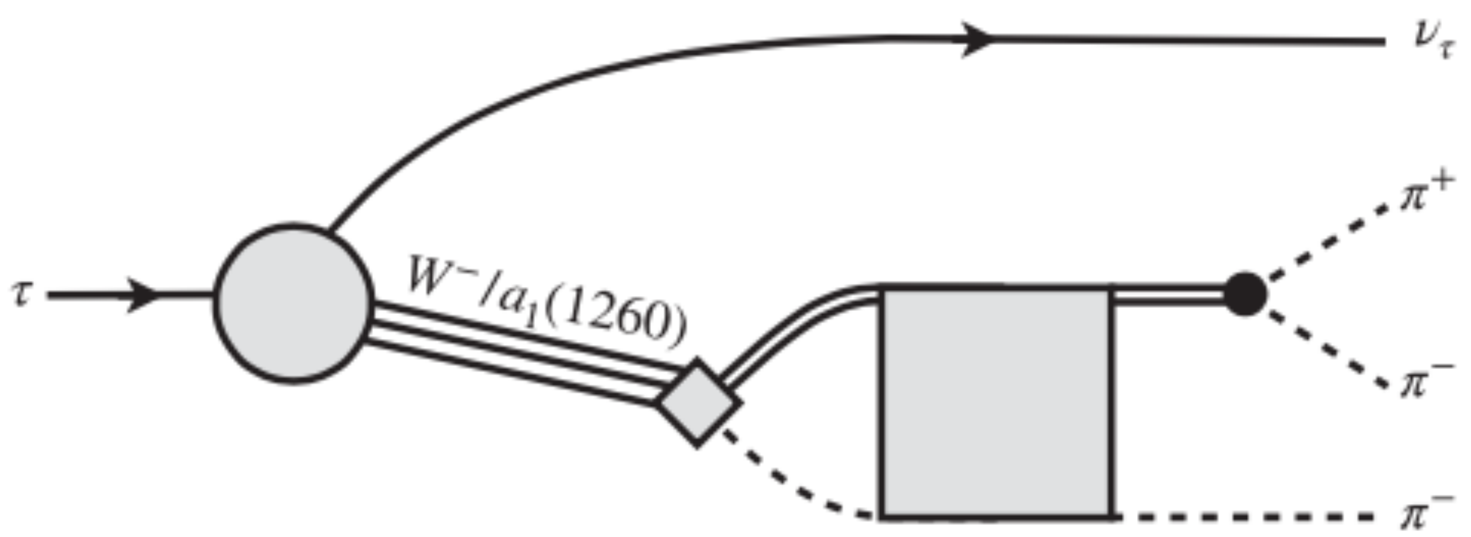
[1] Faddeev, ….
[FIG] Schmid/Ziegelmann Pergamon Press 1974

HADRON SPECTRUM

Impact

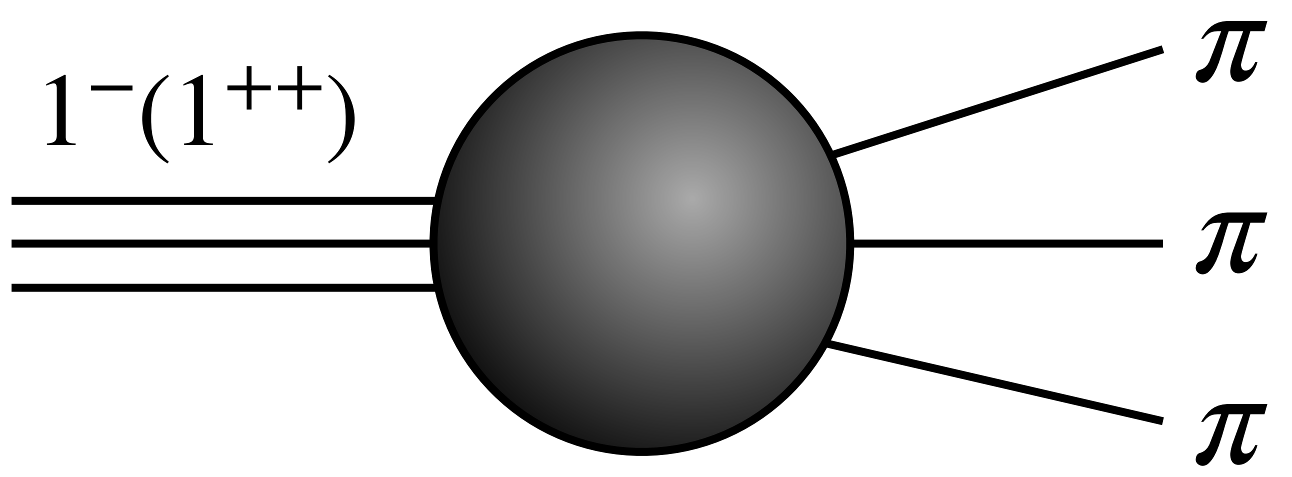
- Many known states have large 3-body content
    - Roper $N(1440)$
    - $X(3872)$
    - $a_1(1260)$, $a_1(1420)$?
- Beyond Standard Model searches (τ-EDM/…)
- Exotic states of matter[1]





[1] Experimental programs: GlueX@JLAB; COMPASS@CERN;
[FIG] Data from Workman et al. (Particle Data Group), Prog. Theor. Exp. Phys. 2022, 083C01 (2022)

RESONANCE PARAMETER

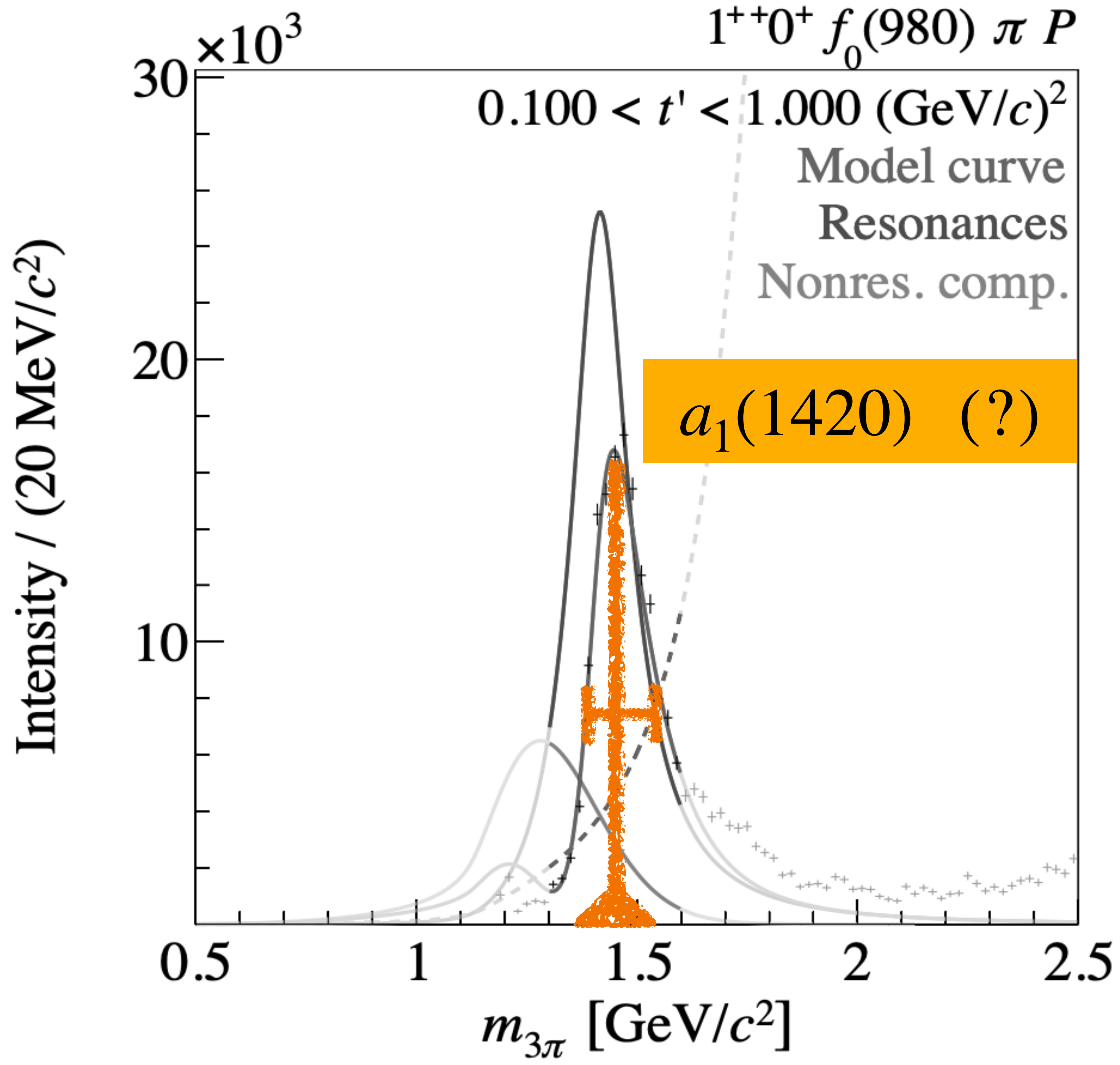


Experimental input

- many high-precision experiments[2] → line-shapes

  resonances ↭ increased interaction rates

  - mod reaction-type
  - mod kinematic singularities[3]

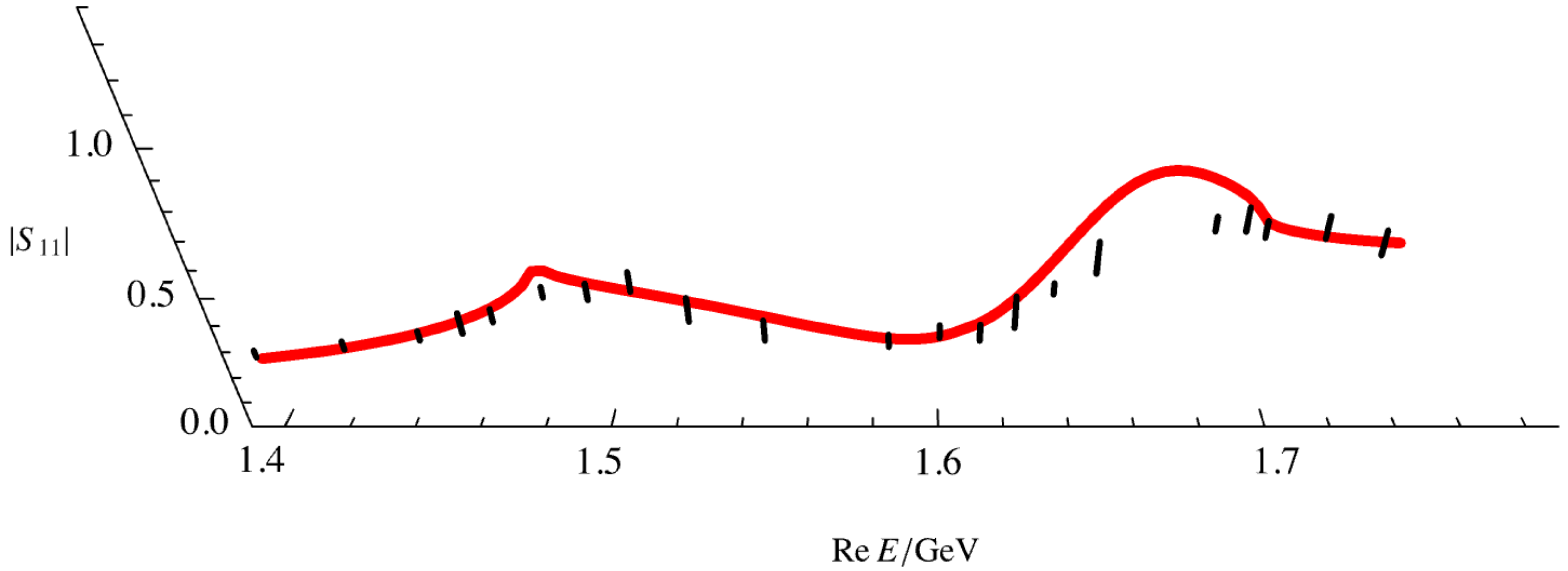


[2] CLAS12, GlueX, …
[3,FIG] [COMPASS] Phys.Rev.Lett. 115 (2015) 8. Review: Ketzer/Grube/Ryabchikov Prog.Part.Nucl.Phys. 113 (2020) 103755

RESONANCE PARAMETER

Universal resonance parameter

- S-matrix theory: ***transition amplitude***
  - Unitarity/Analyticity/Crossing symmetry
  - Poles on unphysical Riemann Sheets
- Boundary ($E \in \mathbb{R}$):
  - Experiment
  - Lattice QCD
  - CHPT



[1] Fig: Data: SAID: Phys. Rev. C 74 (2006) 045205 Model: MM et al. Phys.Rev.D 86 (2012) 094033

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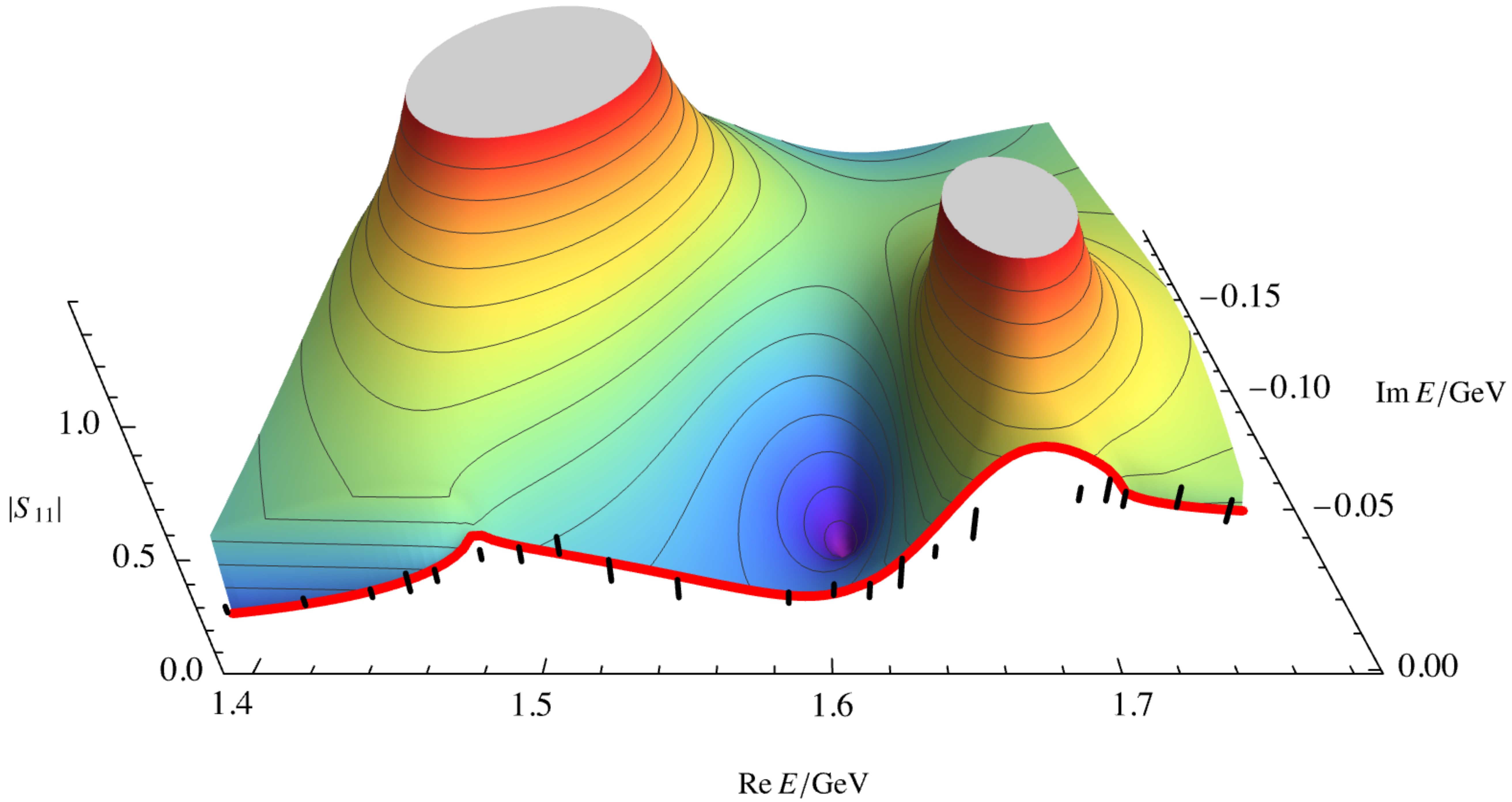

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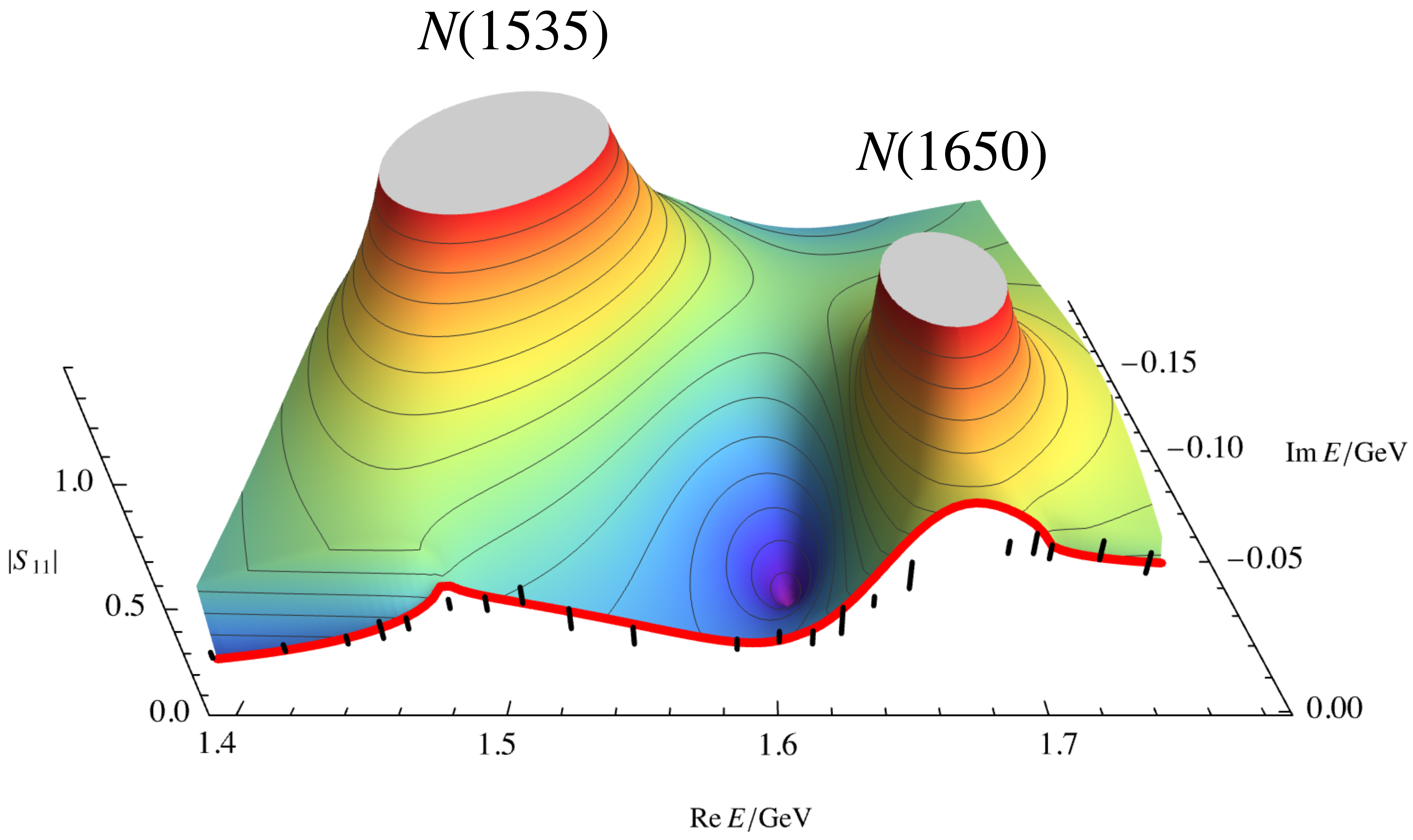

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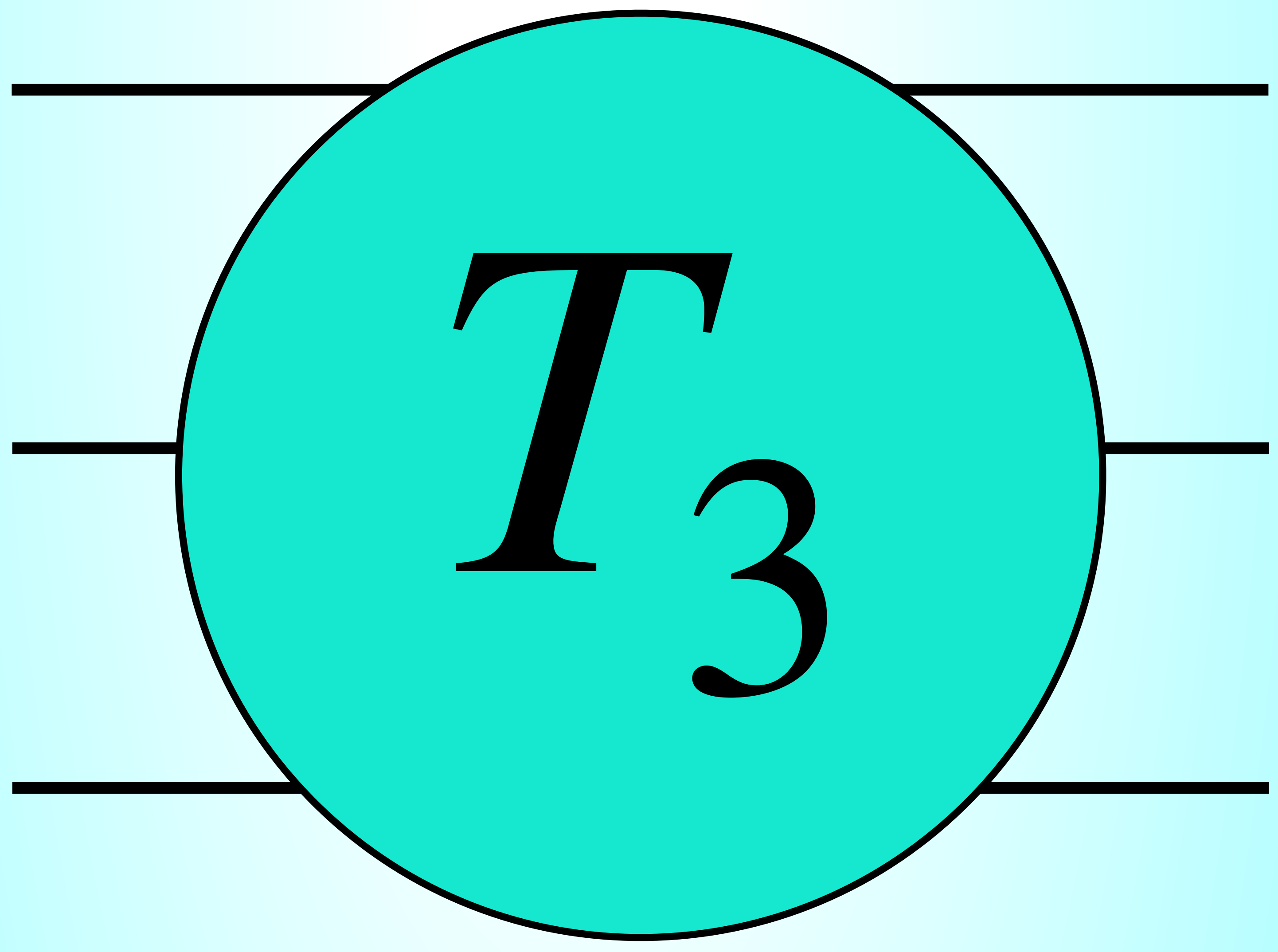
Tridge (Midland, MI/USA)

[1] Willenbrock [arXiv:2203.11056 [hep-ph]], … Eden/Olive/….
Review: MM/Urbach/Meißner *Phys.Rept.* 1001 (2023) 1-66

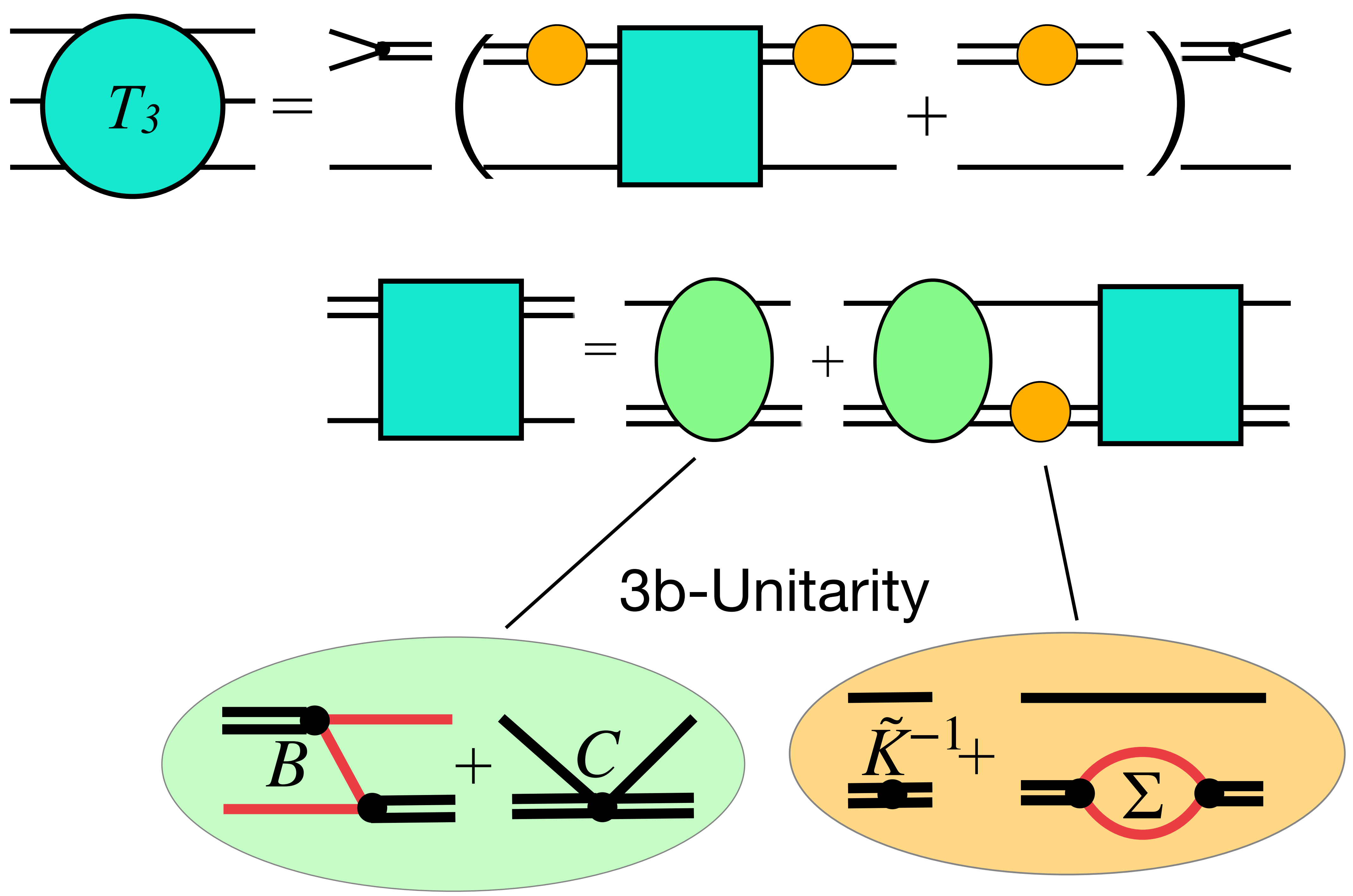
TRANSITION AMPLITUDE



TRANSITION AMPLITUDE

“Infinite Volume Unitarity” — IVU formalism[1]

- Express 3-body through a 2+1 system[2]
  - Unitarity(on-shell configurations): $B, \Sigma \in \mathbb{C}$
  - Dynamics(input): $C, \tilde{K} \in \mathbb{R}$



IVU

$$T^c = B + C + \int \frac{d^3\ell}{(2\pi)^3} \frac{(B+C)}{2E_l} \frac{1}{\tilde{K}_n^{-1} - \Sigma_n} T^c$$

[1] MM/Hu/Döring/Pilloni/Szczepaniak Eur.Phys.J.A 53 (2017)
[2] Related approaches: Hansen/Sharpe(2014)….; Wunderlich et al. JHEP 08 (2019); Jackura et al. Eur.Phys.J.C 79 (2019);

HILBERT'S HOTEL

"Infinite Volume Unitarity" — IVU formalism

- Analytic structure of the one-particle exchange
  - Left-hand cuts[1] – $T_{cc}(3875)$ etc..
  - Landau singularities
    - Triangles[2] + Boxes + Boxes+…[3]

https://www.ias.edu/ideas/2016/pires-hilbert-hotel

[1] Du et al. Phys.Rev.Lett. 131 (2023) 13; Hansen et al. 2401.06609 [hep-lat]
[2] Korpa/Lutz/Guo/Heo *Phys.Rev.D* 107 (2023) 3; Isken et al. 2309.09695; … Ketzer/Mikhashenko/Aceti/Dai/Oset/Bayar/Guo…
[3] Sakhtivasan/MM in preparation

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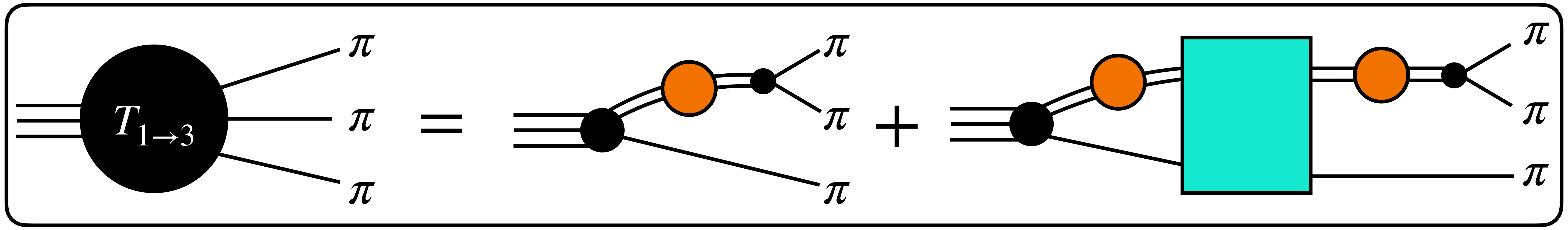

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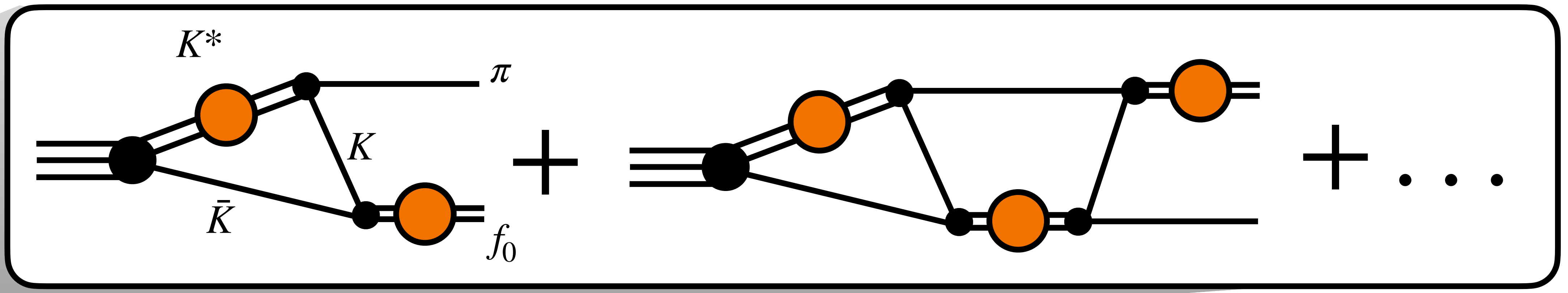

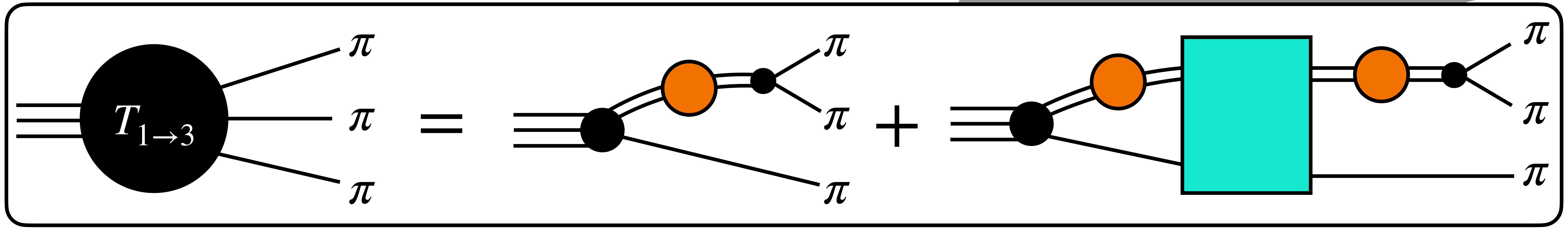

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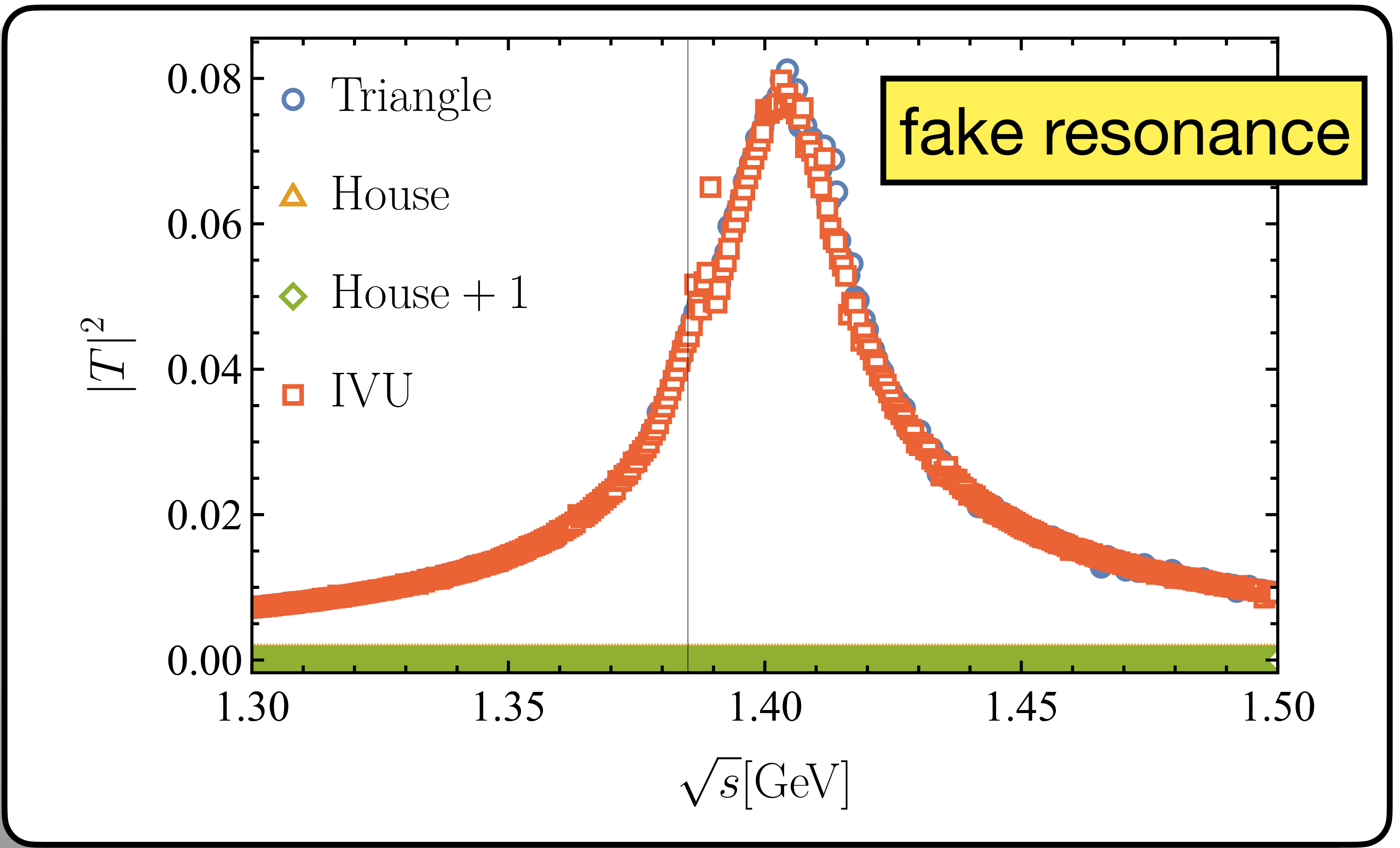

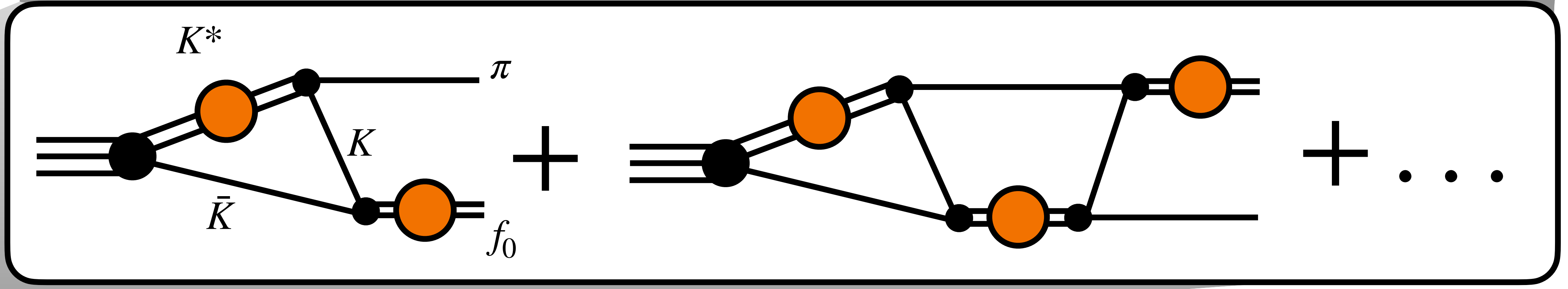

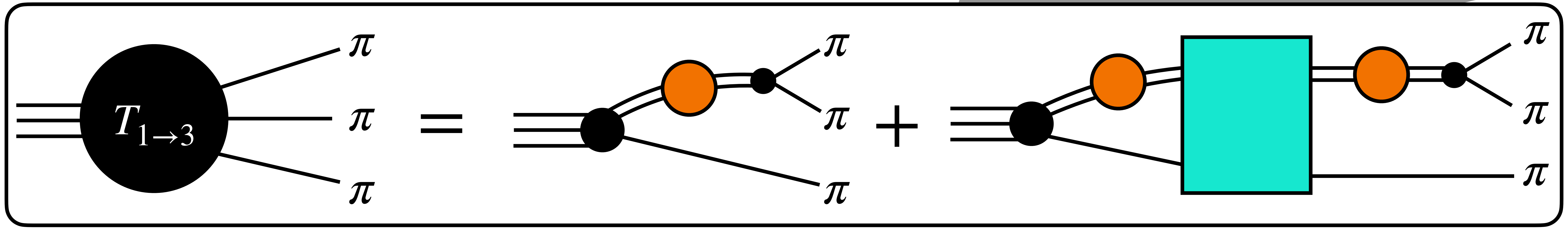

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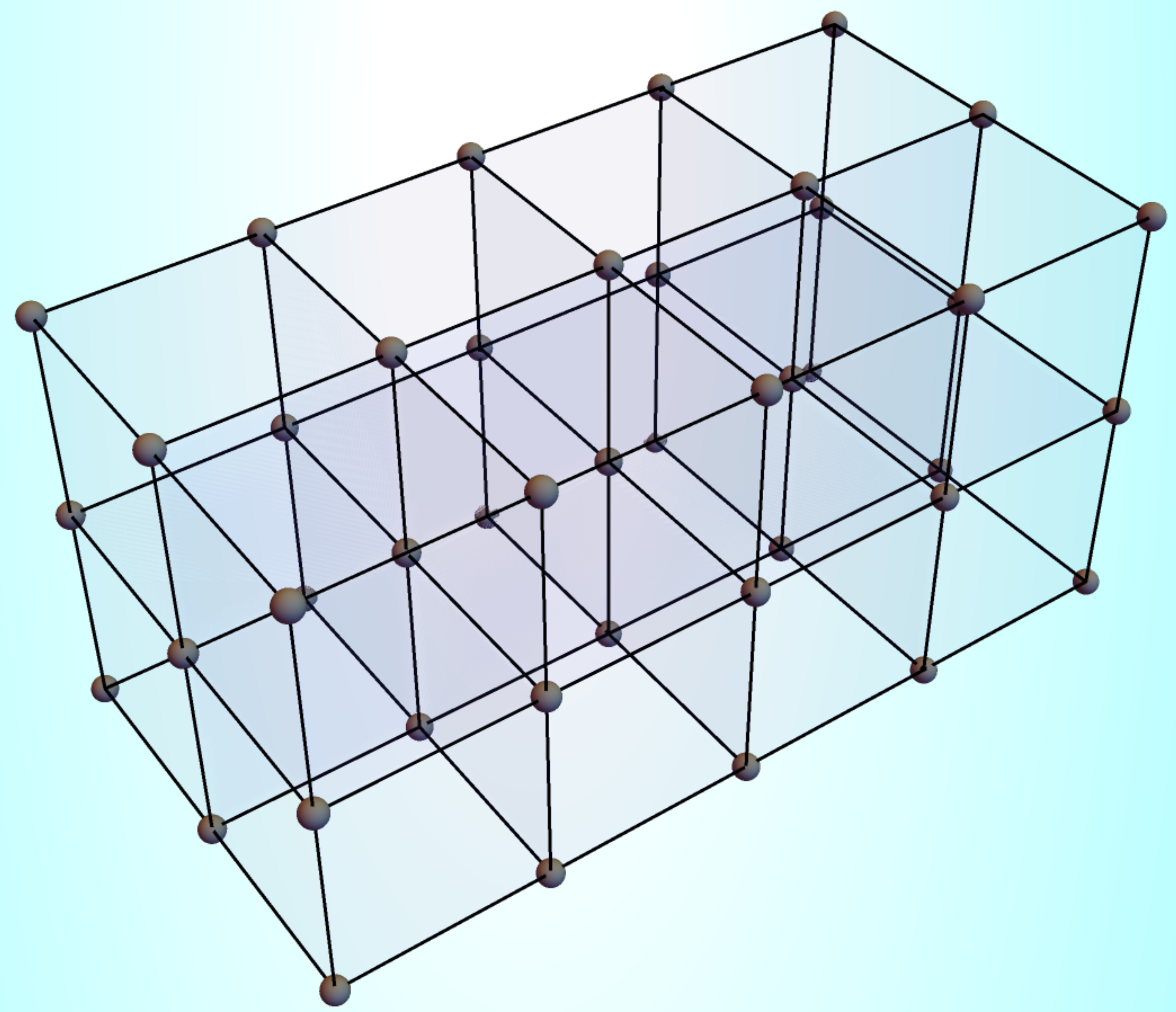
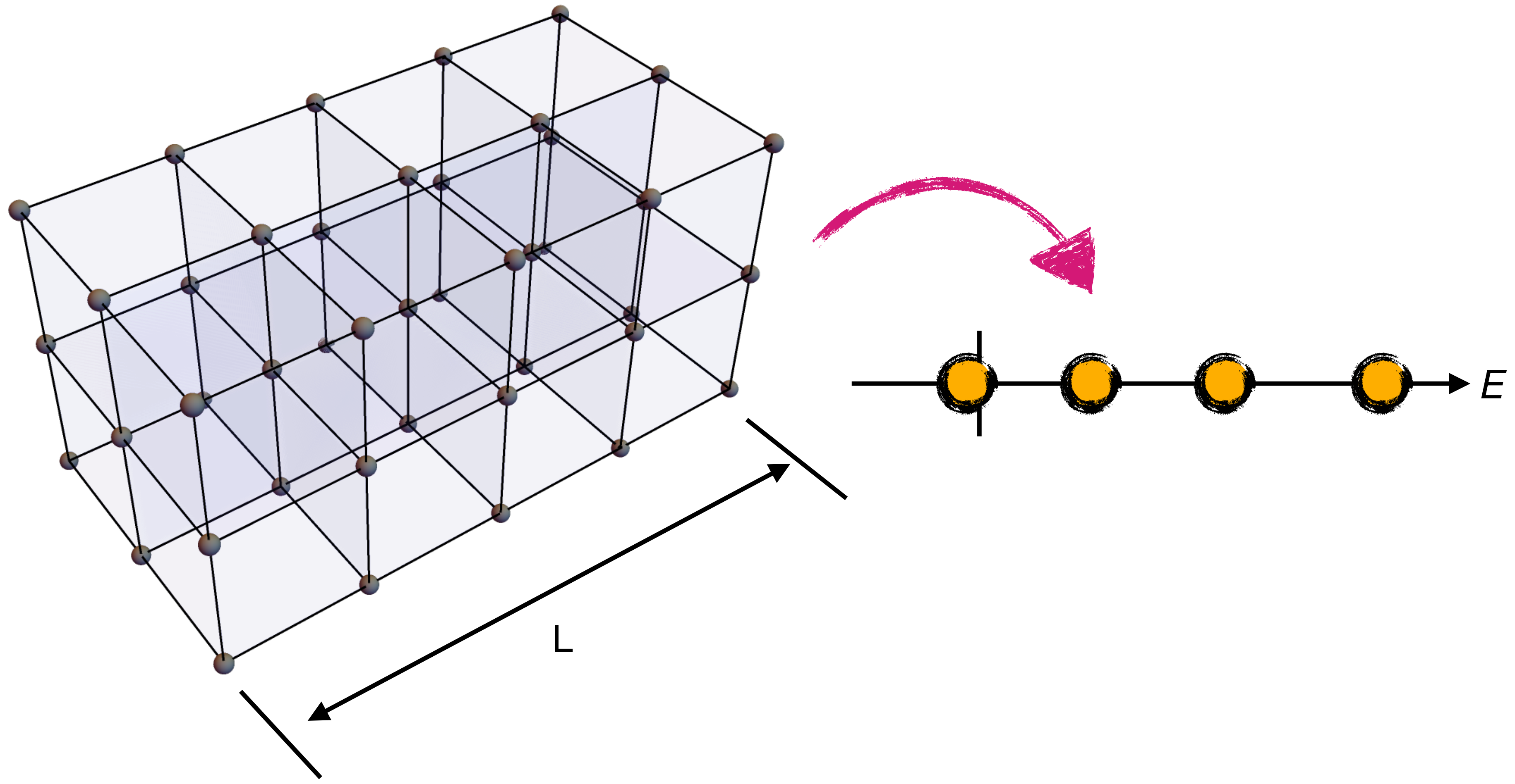
APPLICATIONS TO LATTICE QCD



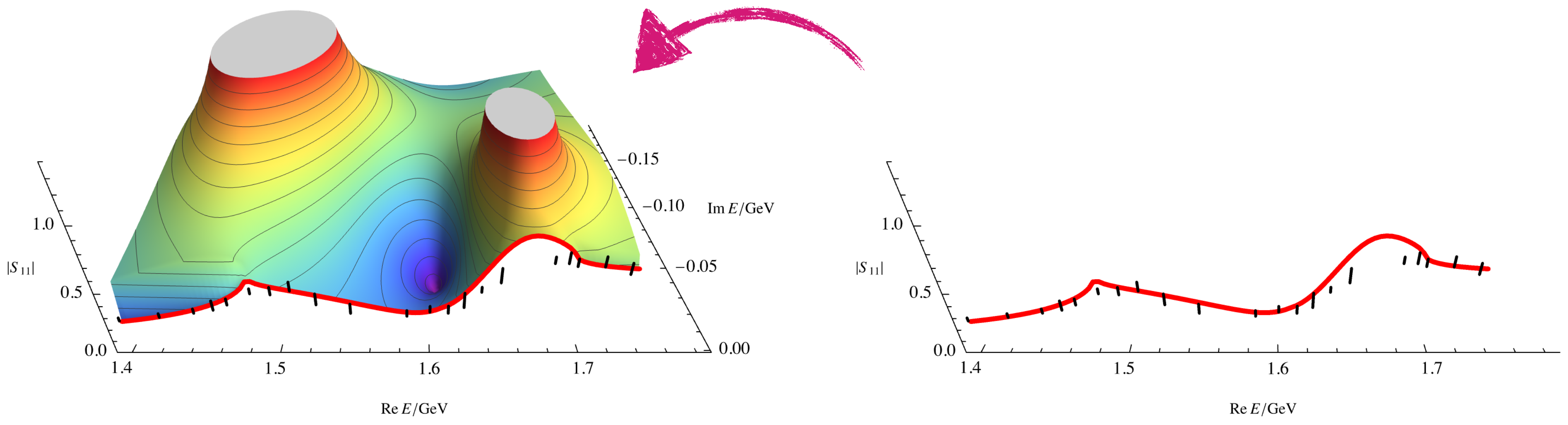
FINITE-VOLUME SPECTRUM

Lattice QCD: numerical access to QCD Green's functions:

Euclidean space-time / unphysical pion mass / **finite-volume**



S-matrix, phenomenology, experiment…



[1] Lüscher, Gottlieb, Rummukainen, Feng, Li, Döring, Briceño, Meißner, Rusetsky, Hansen, MM, Blanton, …
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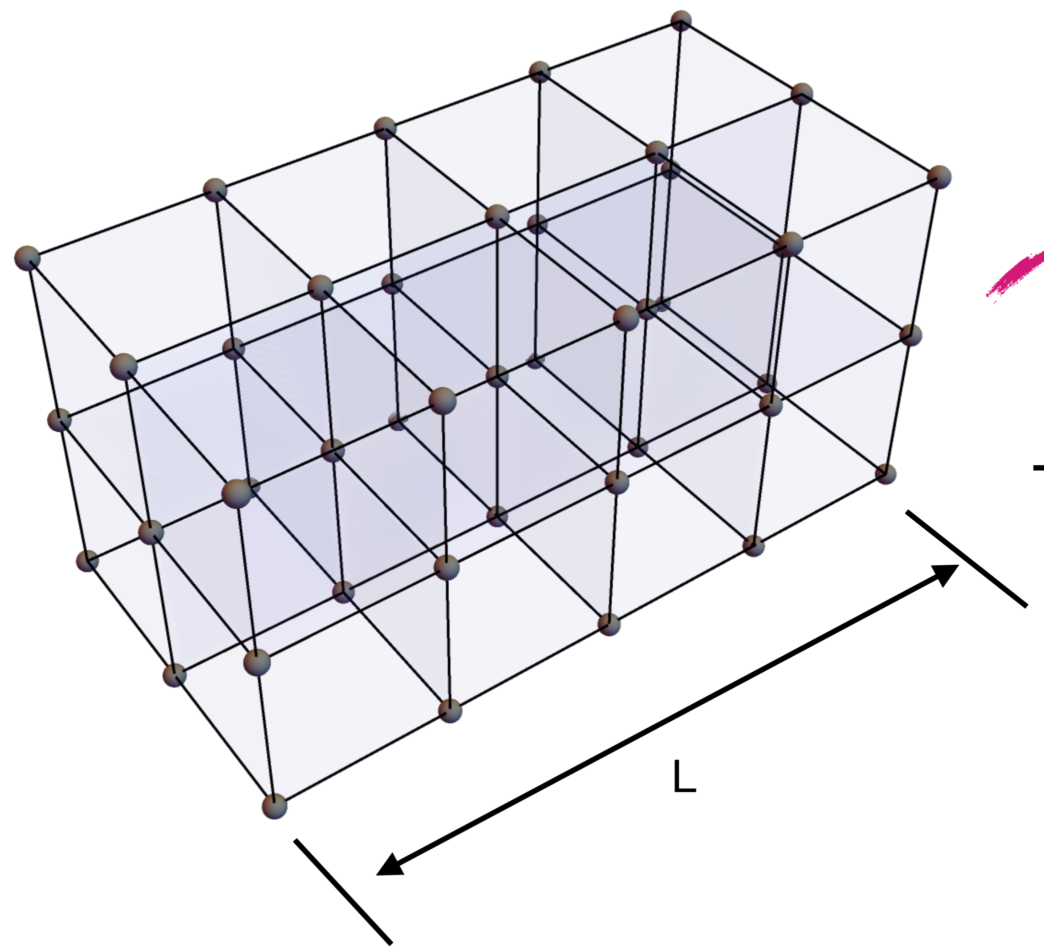
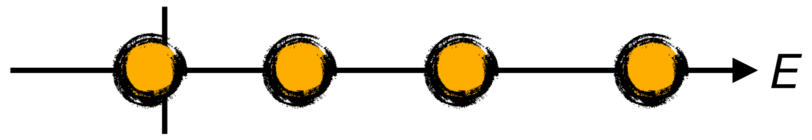
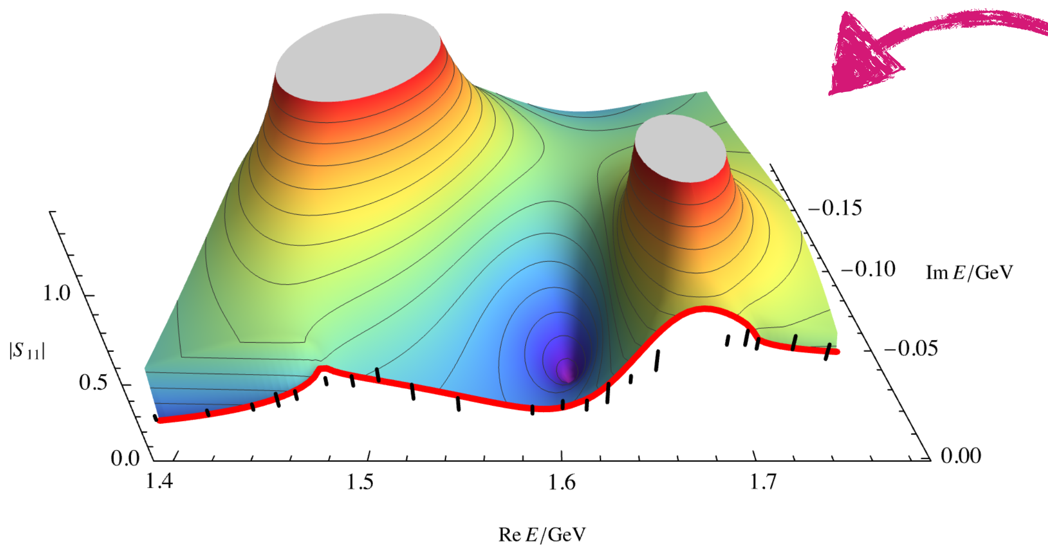


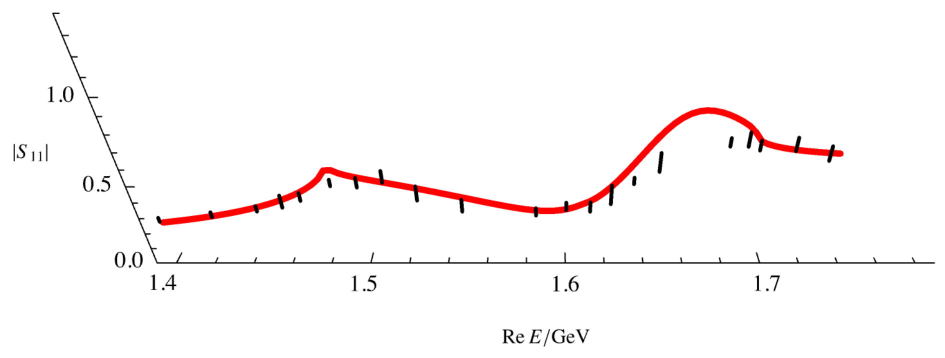


"Finite Volume Unitarity" — FVU formalism

- On-shell particles "feel" the box-size
- Three-body **quantization condition**

S-matrix, phenomenology, experiment…





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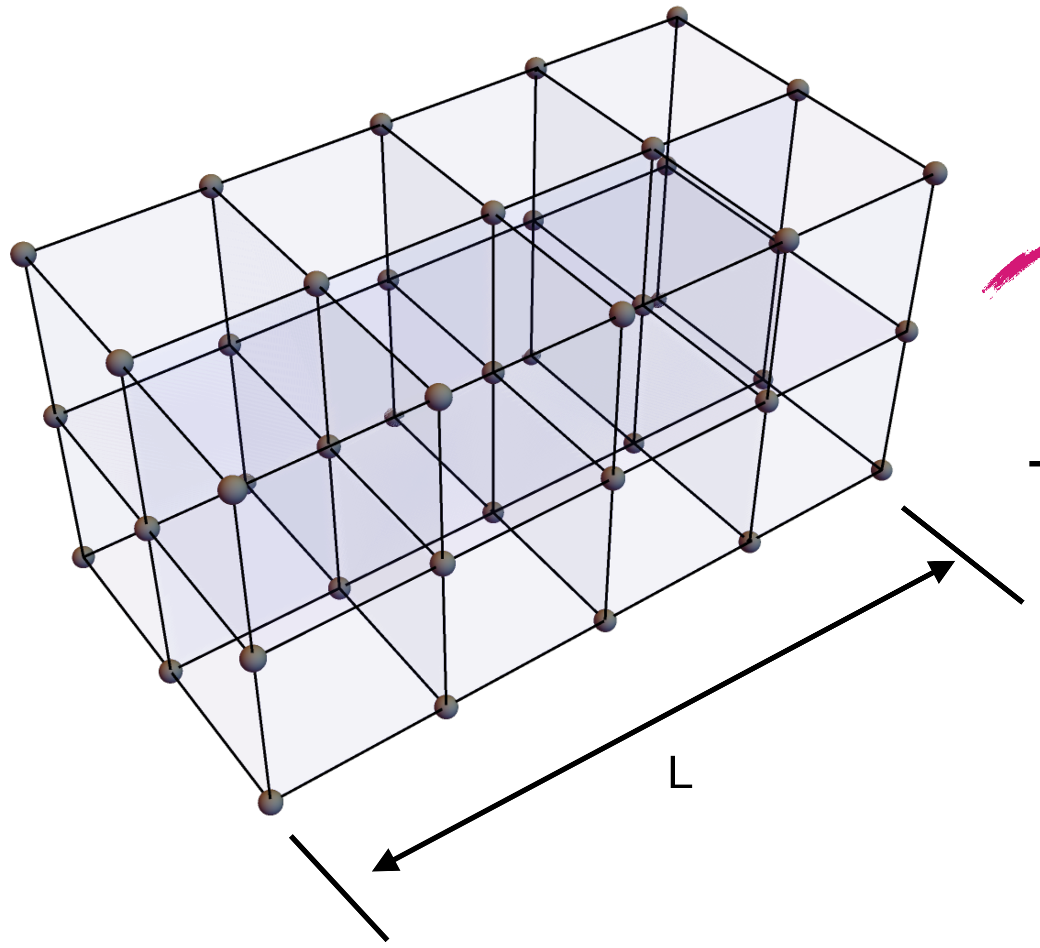

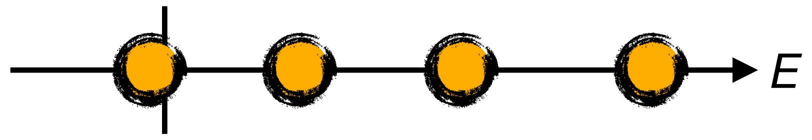

FVU

$$\det\left[\,2L^3E_{\mathbf{p}}\left(\tilde{K}_2^{-1}-\Sigma_2^L\right)-B-C\,\right]^{T_{1g}}$$

"Finite Volume Unitarity" — FVU formalism

- On-shell particles "feel" the box-size
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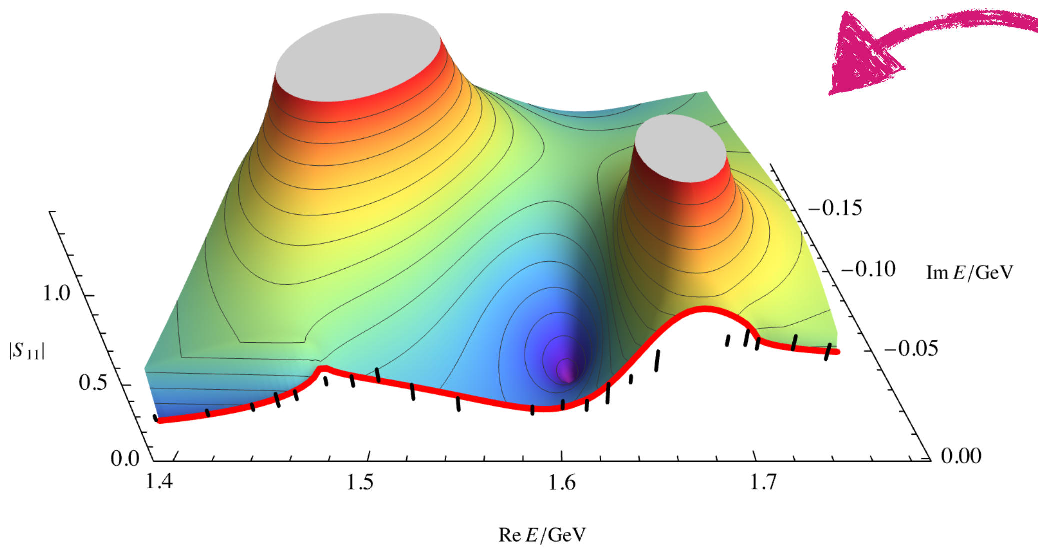

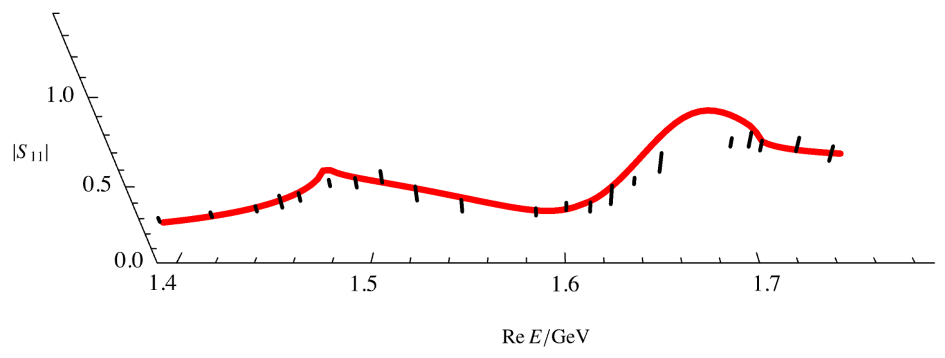

IVU

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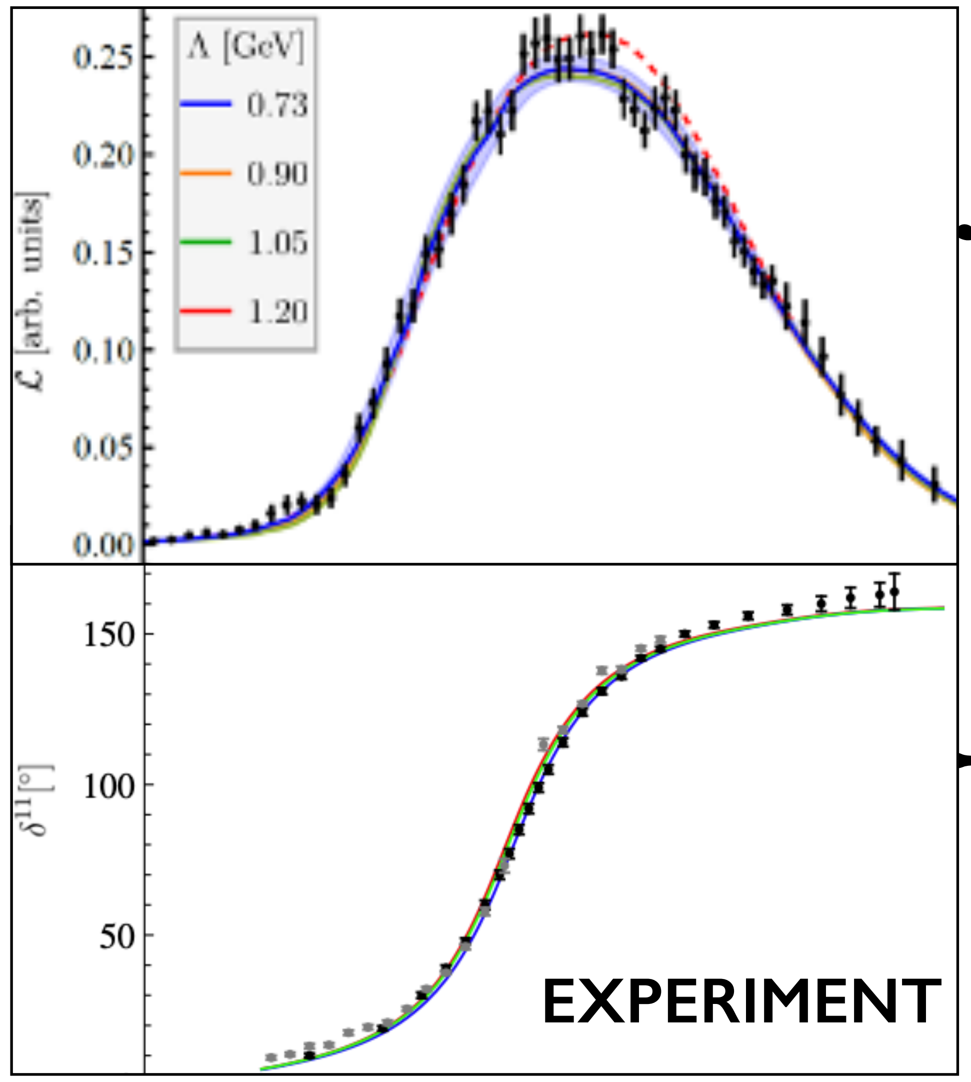
BLUEPRINT — $a_1(1260)$

INPUT[1]



TRANSITION AMPLITUDES

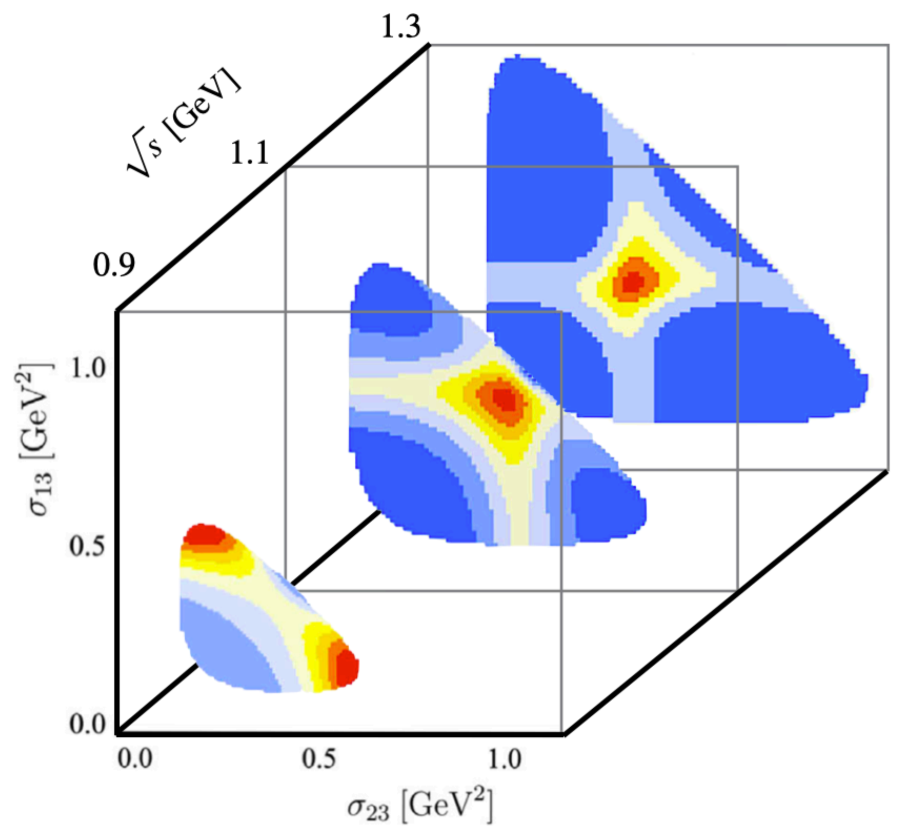
IVU

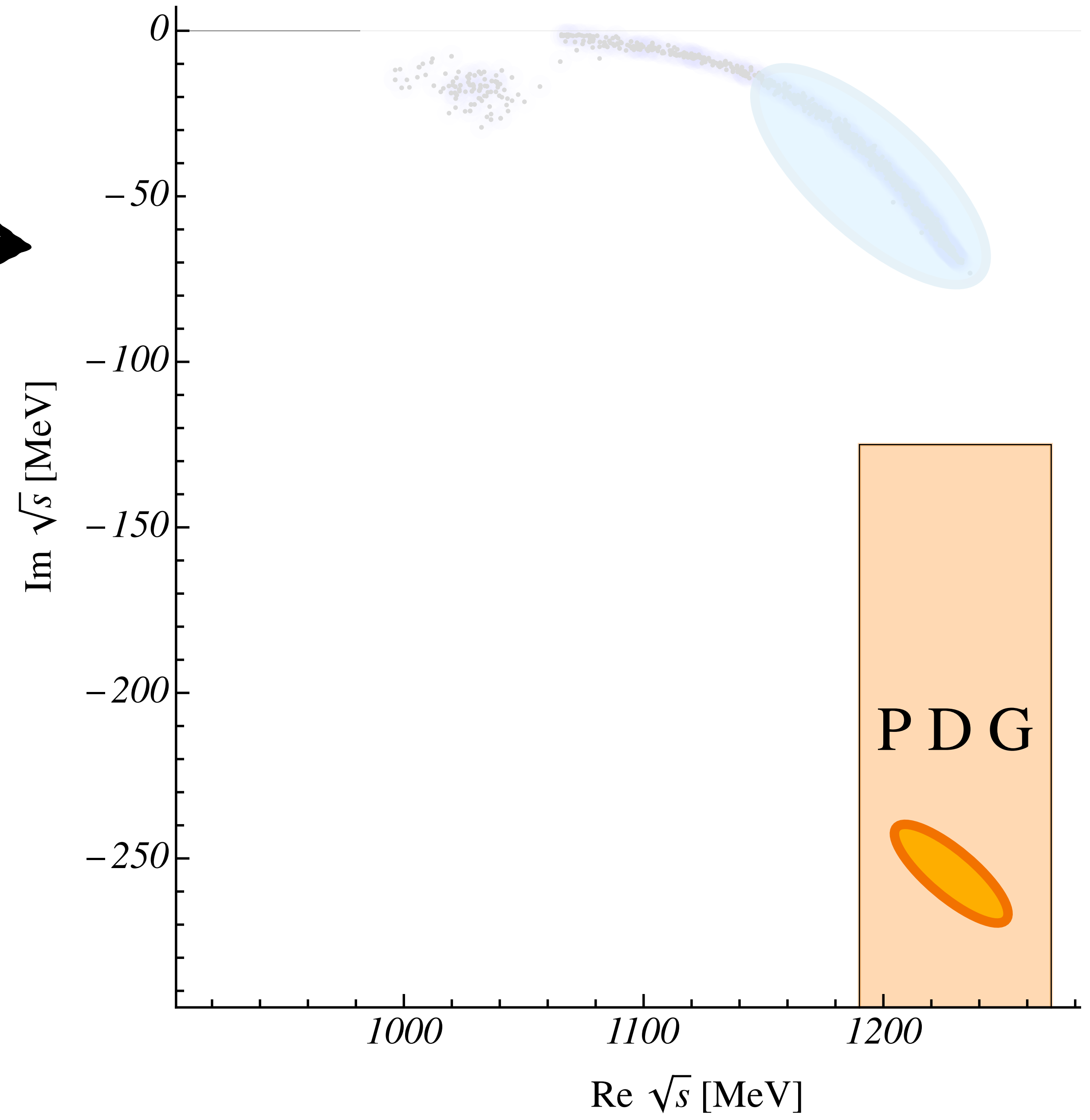
$$T^c = B + C + \int \frac{d^3\ell}{(2\pi)^3} \frac{(B+C)}{2E_l} \frac{1}{\tilde{K}_n^{-1} - \Sigma_n} T^c$$

FVU

$$\det\left[2L^3E_{\mathbf{p}}\left(\tilde{K}_2^{-1} - \Sigma_2^L\right) - B - C\right]^{T_{1g}}$$

OUTPUT[2]

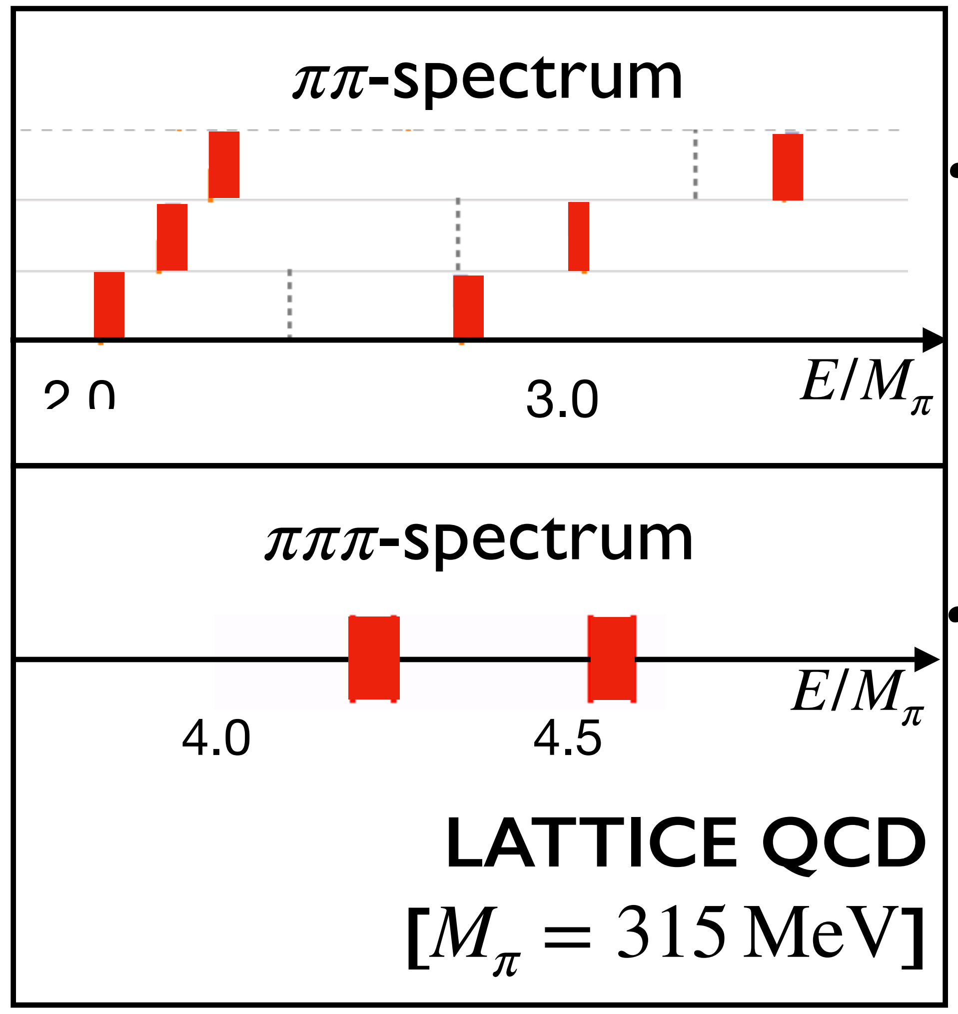




[1] Schael [ALEPH] Phys.Rept. 421 (2005); Nucl.Phys.B 79; Phys.Rev.D 7; [GWQCD] PRD94(2016) PRD98 (2018) PRD 100(2019)
[2] Sadasivan/MM/Döring/Alexandru/Culver/Lee Phys.Rev.D 101 (2020); MM/Culver/Sadasivan/Brett/Döring/Alexandru/Lee [GWQCD] PRL 127 (2021)
other phenomenological determinations: JPAC/….

BLUEPRINT — $a_1(1260)$

INPUT[1] **TRANSITION AMPLITUDES** **OUTPUT[2]**



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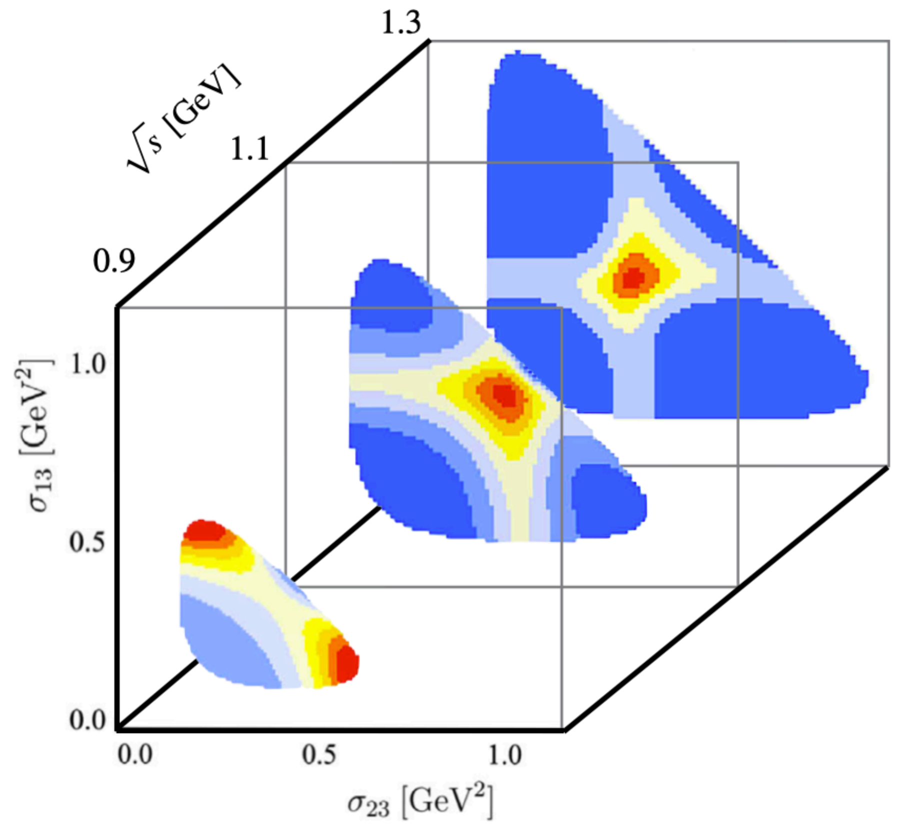

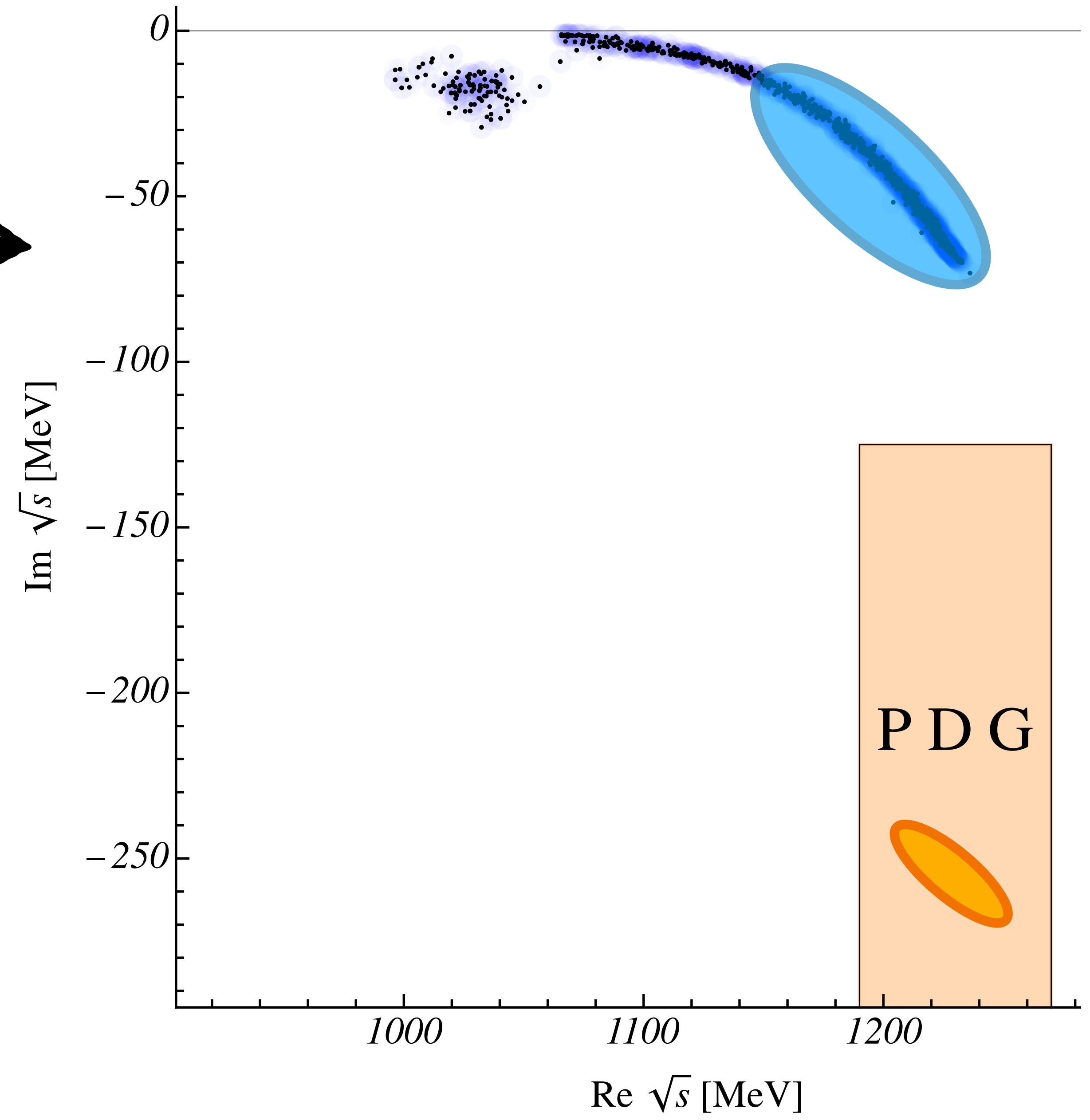

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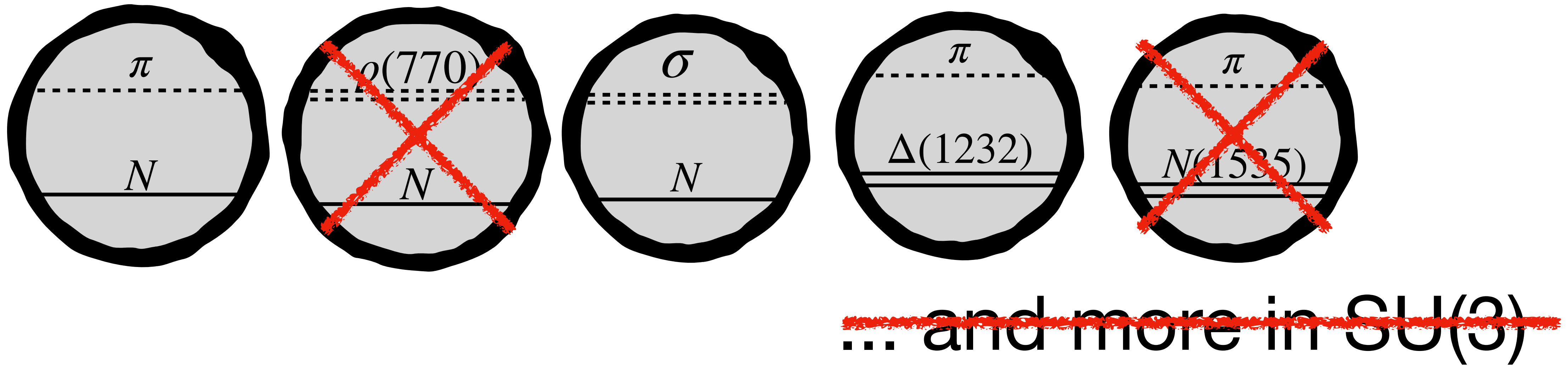
other phenomenological determinations: JPAC/….

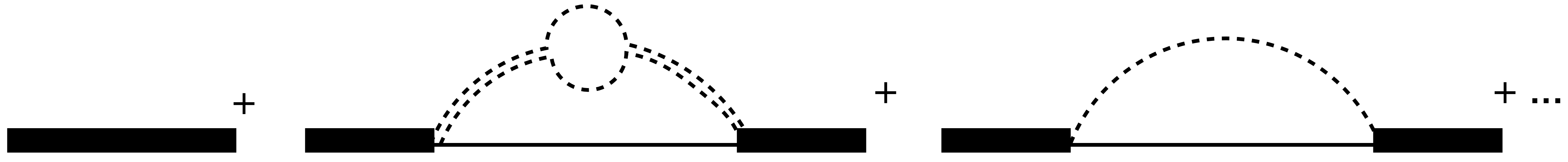
ROPER $N(1440)$ — FINITE VOLUME



Simplified pilot study[1]

- self-energy formalism via particle-dimer Lagrangian

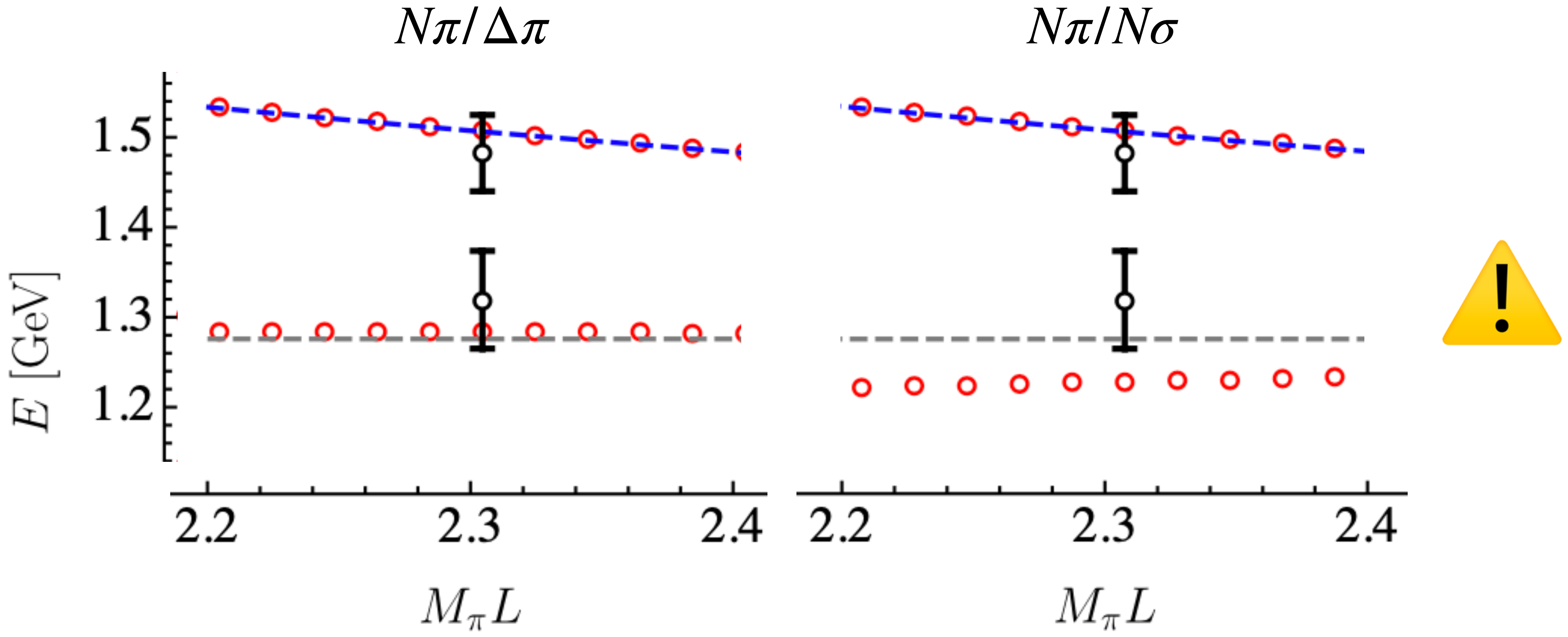
⚠️ no particle-exchange diagrams



Predict finite-volume spectrum for fixed parameters

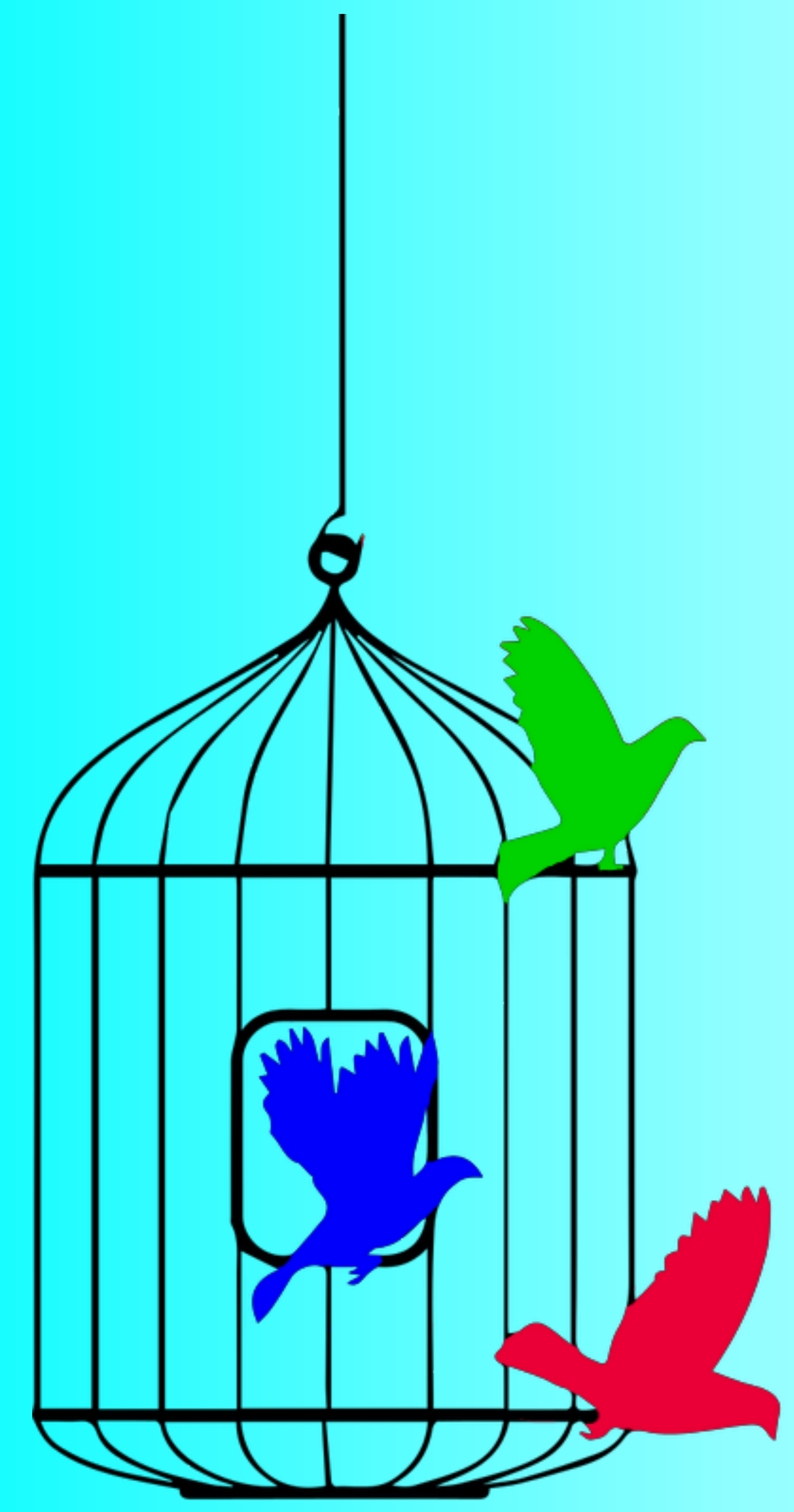
→ energy shifts very small (opposing effects of $N\boldsymbol{\sigma}$ and $\boldsymbol{\Delta\pi}$ channels)

→ phenomenological input necessary



[1] Severt, MM, Ulf-G.Meißner *JHEP* 04 (2023) 100
[2] Lattice values (black dots) Lang et al. Phys.Rev.D 95 (2017) 1

SUMMARY



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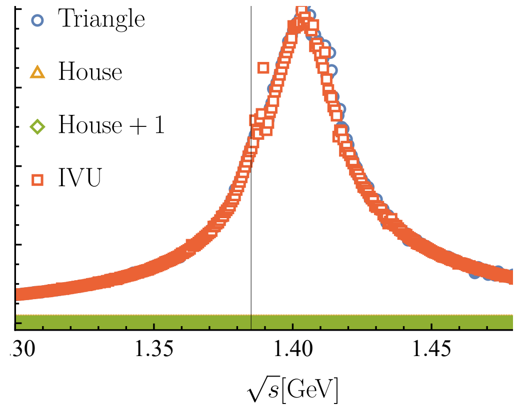
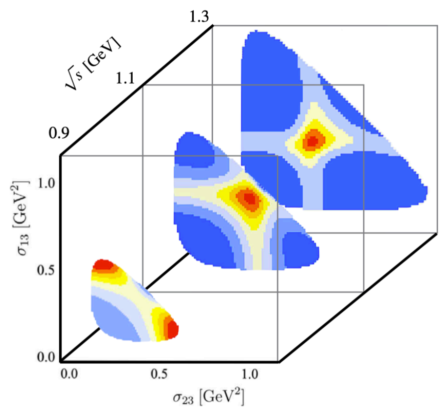


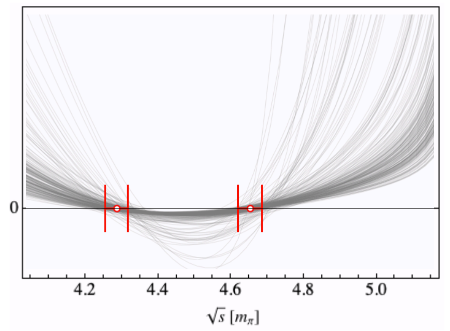


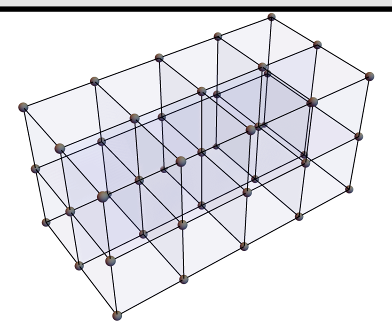
Infinite volume three-body formalism

- Unitarity induced analytic structure
- universal resonance parameter
- singularity structure/Landau singularities

FVU

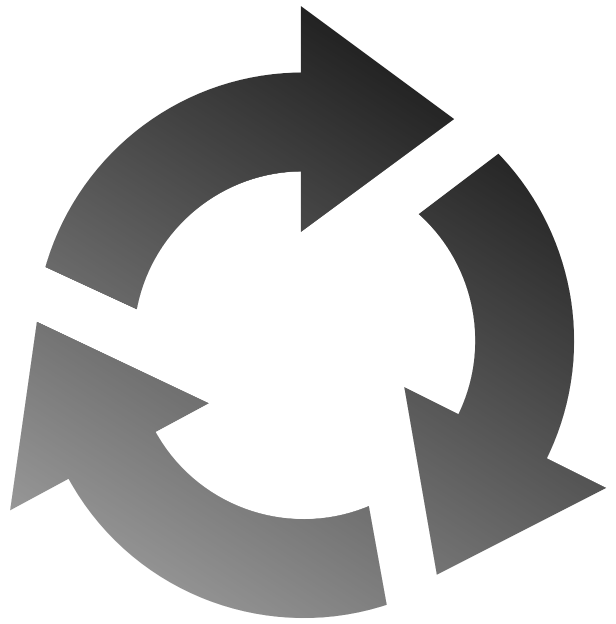
$$\det\left[\,2L^3E_{\mathbf{p}}\left(\tilde{K}_2^{-1}-\Sigma_2^L\right)-B-C\,\right]^{T_{1g}}$$





Finite-volume three-body formalism FVU

- 3b quantization condition
- several applications
- first chiral trajectories of 3b-resonances



OUTLOOK

- $\pi\pi N$ content of Roper-resonance

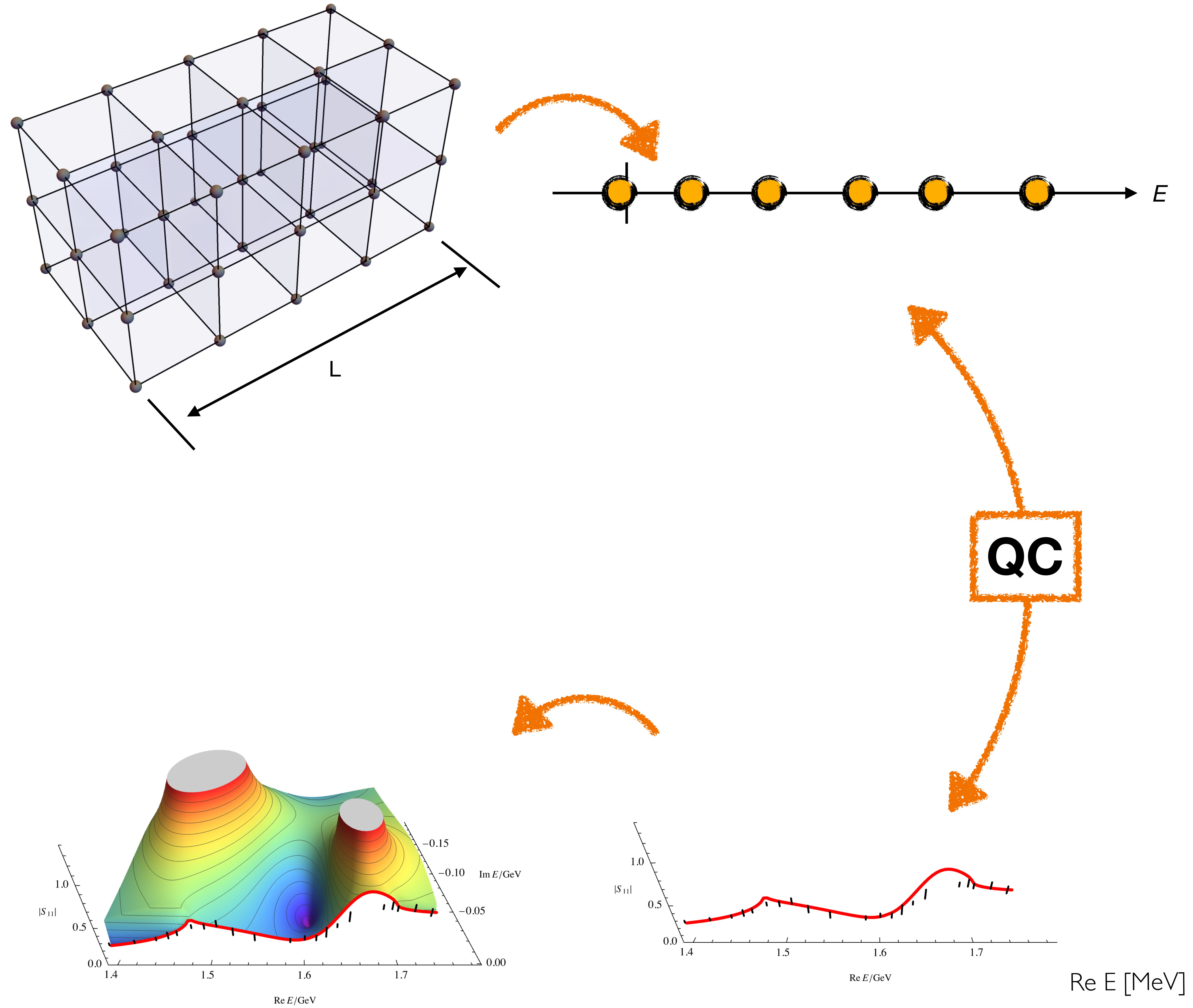
  … connections to DCC global studies
- $\pi\pi\Lambda$ and strangeness resonances (?)
- $\bar{K}d$ scattering
- …

THANK YOU

FINITE-VOLUME SPECTRUM

Lattice QCD: numerical access to QCD Green's functions:

Euclidean space-time / unphysical pion mass / **finite-volume**



“Finite Volume Unitarity” — FVU formalism

- On-shell particles “feel” the box-size
- Three-body quantization condition

FVU

$$\det\left[\,2L^3E_{\mathbf{p}}\left(\tilde{K}_2^{-1}-\Sigma_2^L\right)-B-C\,\right]^{T_{1g}}$$

[1] Lüscher, Gottlieb, Rummukainen, Feng, Li, Döring, Briceño, Meißner, Rusetsky, Hansen, MM, Blanton, …
Reviews: Briceno/Dudek/Young (2017) Rev.Mod.Phys. 90 (2018) 2 Hansen/Sharpe Ann.Rev.Nucl.Part.Sci. 69 (2019); MM/Doring/Rusetsky Eur.Phys.J.ST 230 (2021);

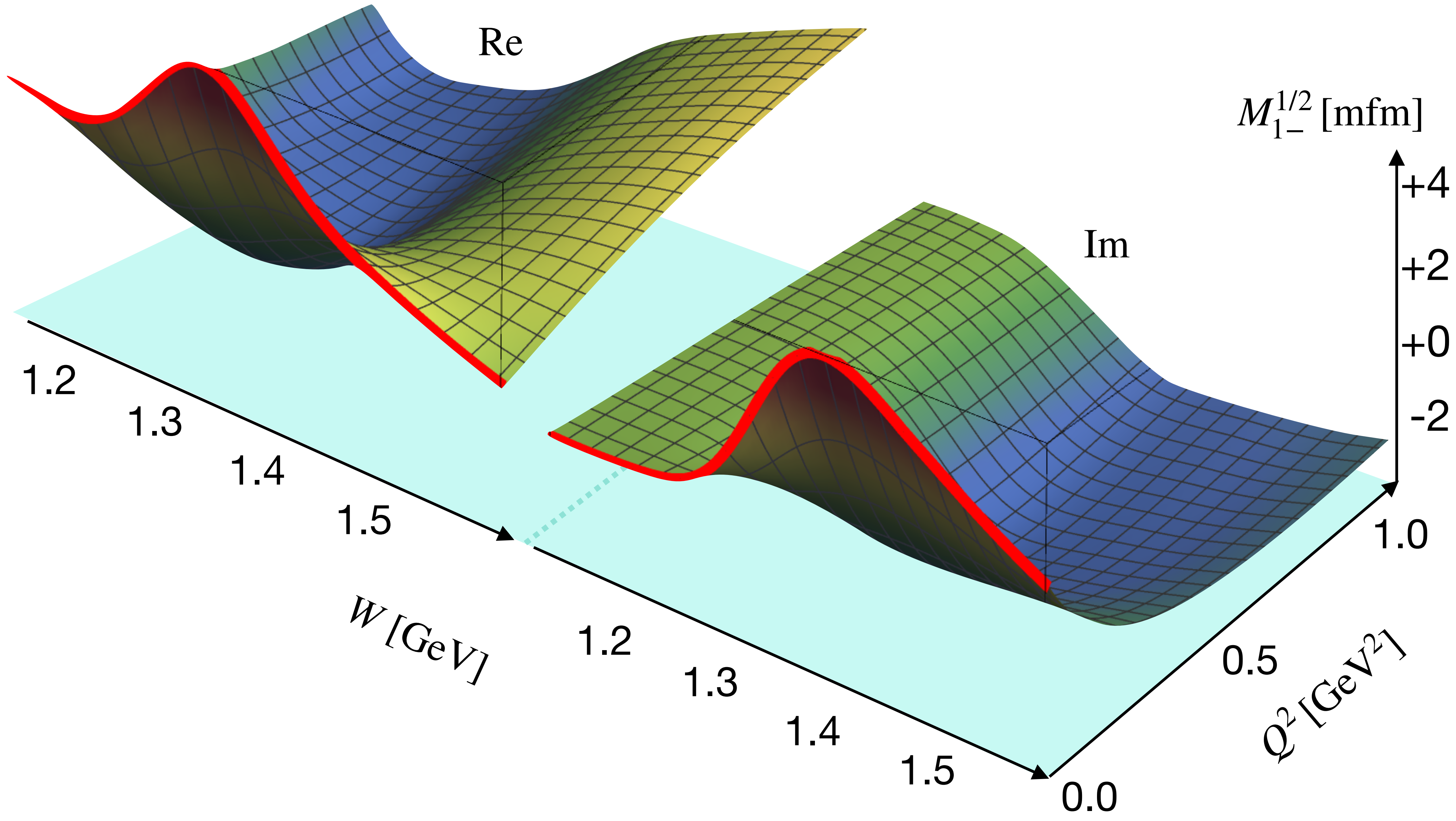
ROPER $N(1440)$ — PHENOMENOLOGY

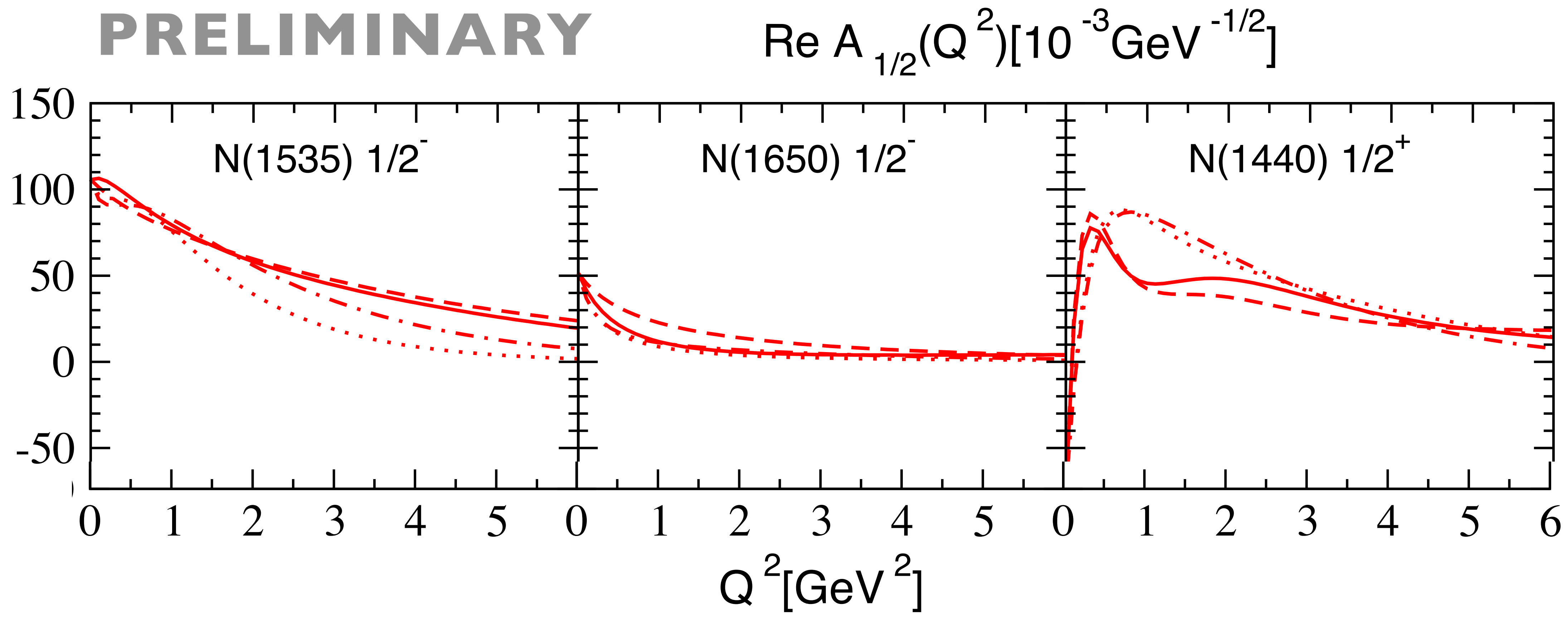
Global analysis (bird's view)

- many experimental data & ongoing experiments
  - @MAMI, @ELSA, @JLAB,…
  - $\gamma N \to \pi(\pi)N, \eta N, K\Lambda\ldots$
- Jülich-Bonn-Washington[1,2] DCC

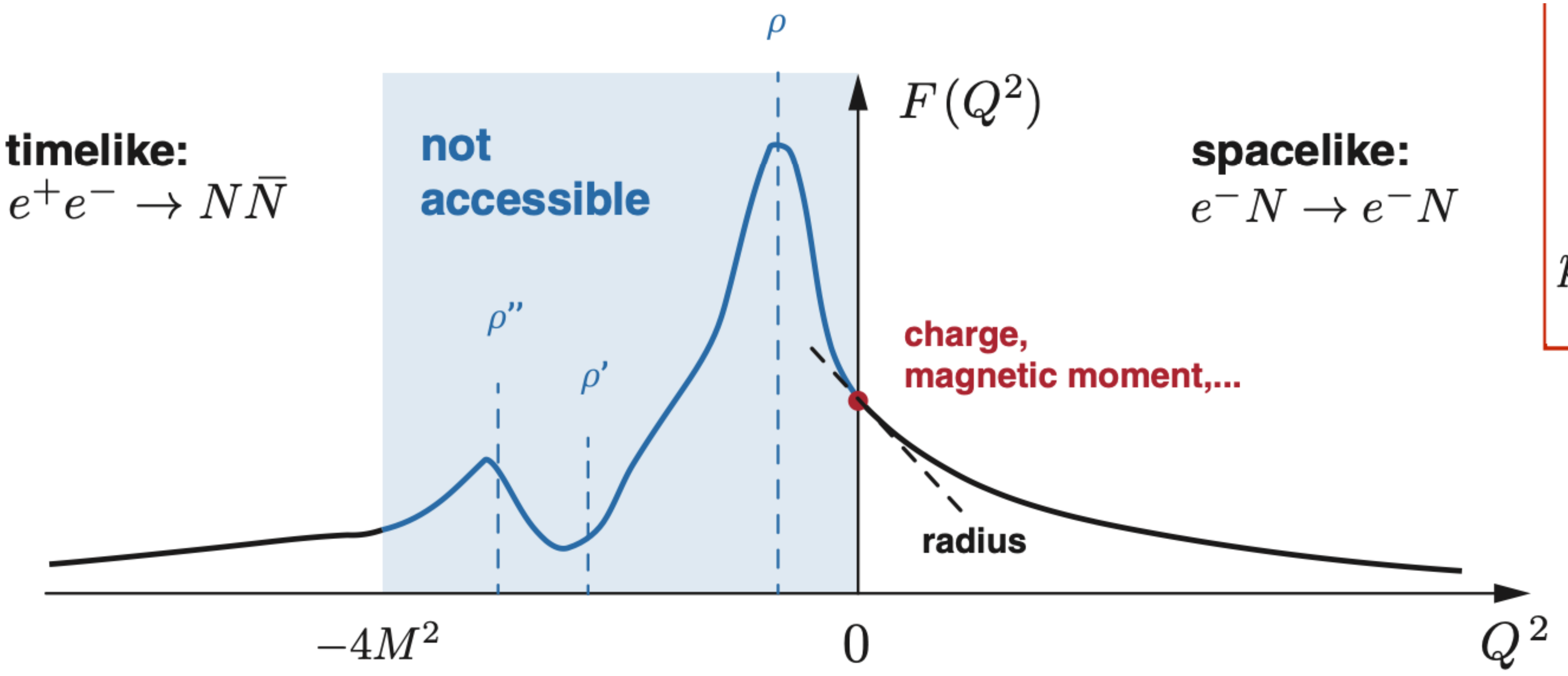
  → Roper has very unusual $f(W, Q^2)$: $\pi\pi N$ effect(?)

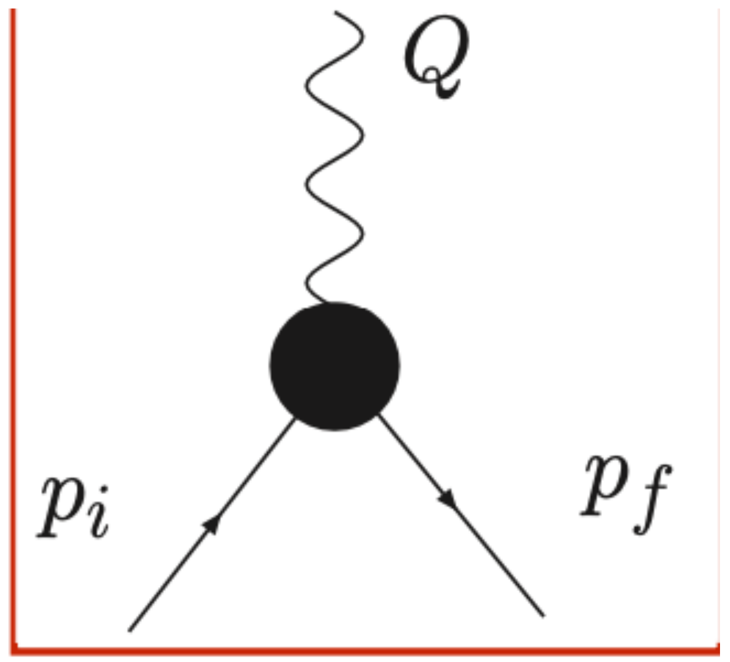
  → Transition form-factors[3]





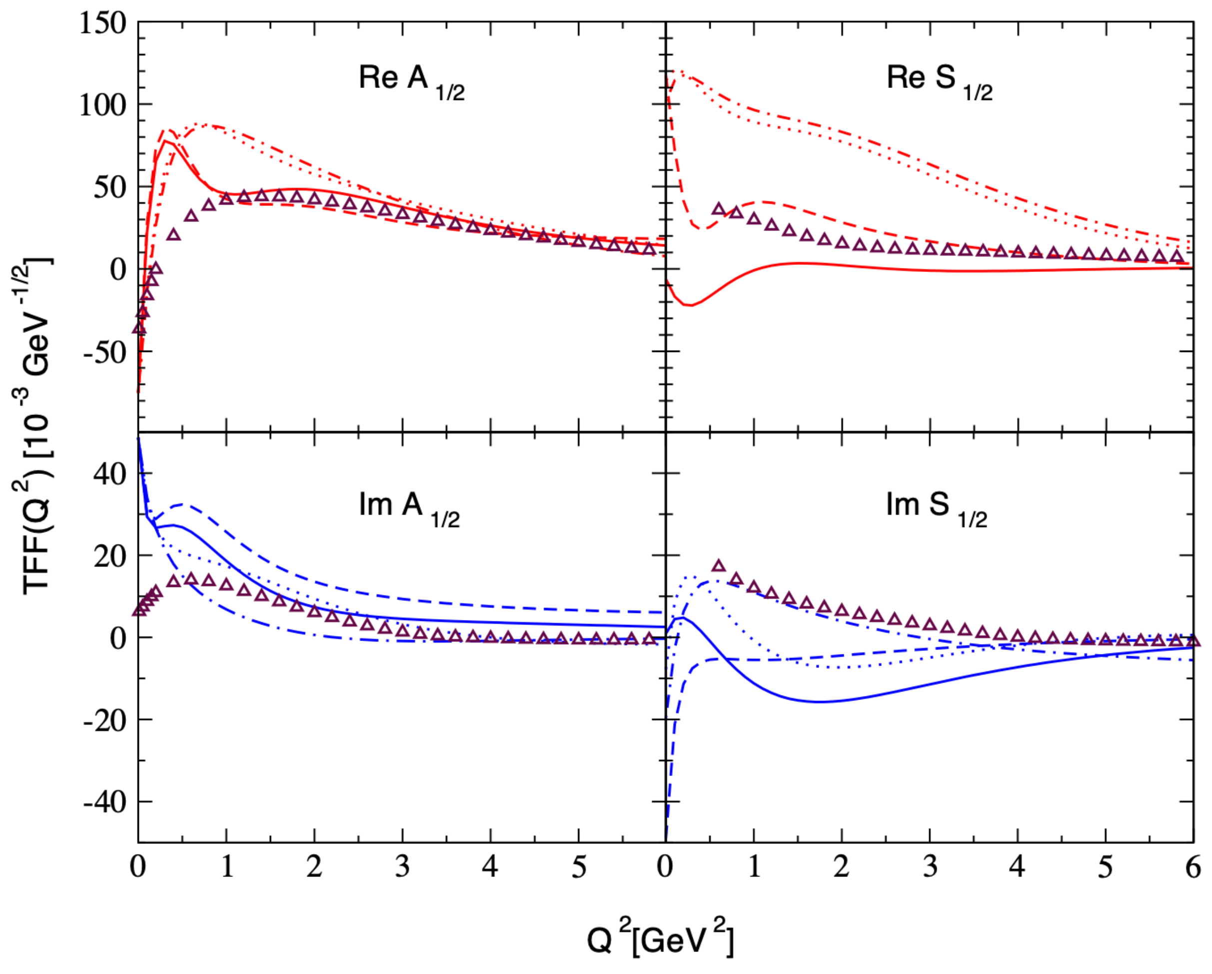
[1] [JBW] MM et al. Phys.Rev.C 103 (2021) 6; Phys.Rev.C 106 (2022) 015201; Eur.Phys.J.A 59 (2023) 12; jbw.phys.gwu.edu/
[2] Related approaches MAID/SAID/Gent/ANLOsaka
[3] Wang/MM/… in progress





Gernot Eichmann

PRELIMINARY



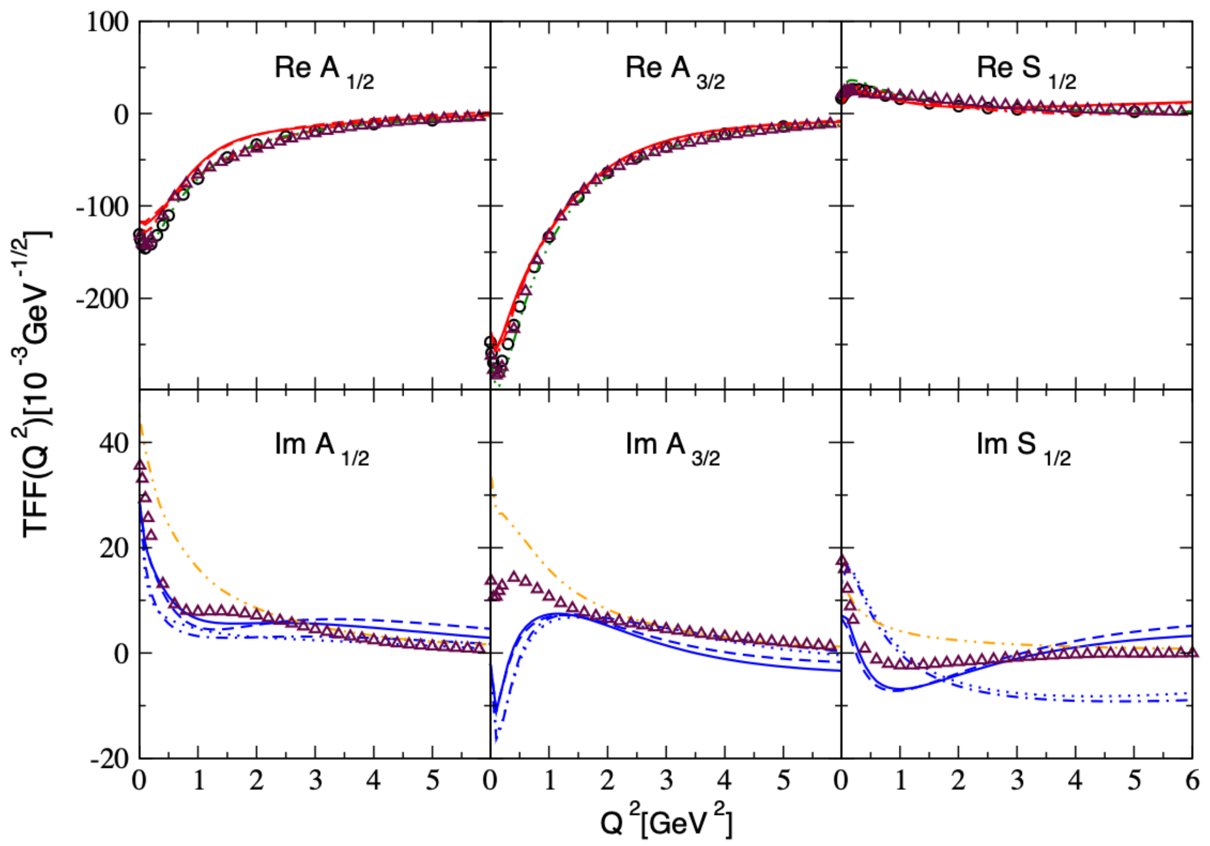
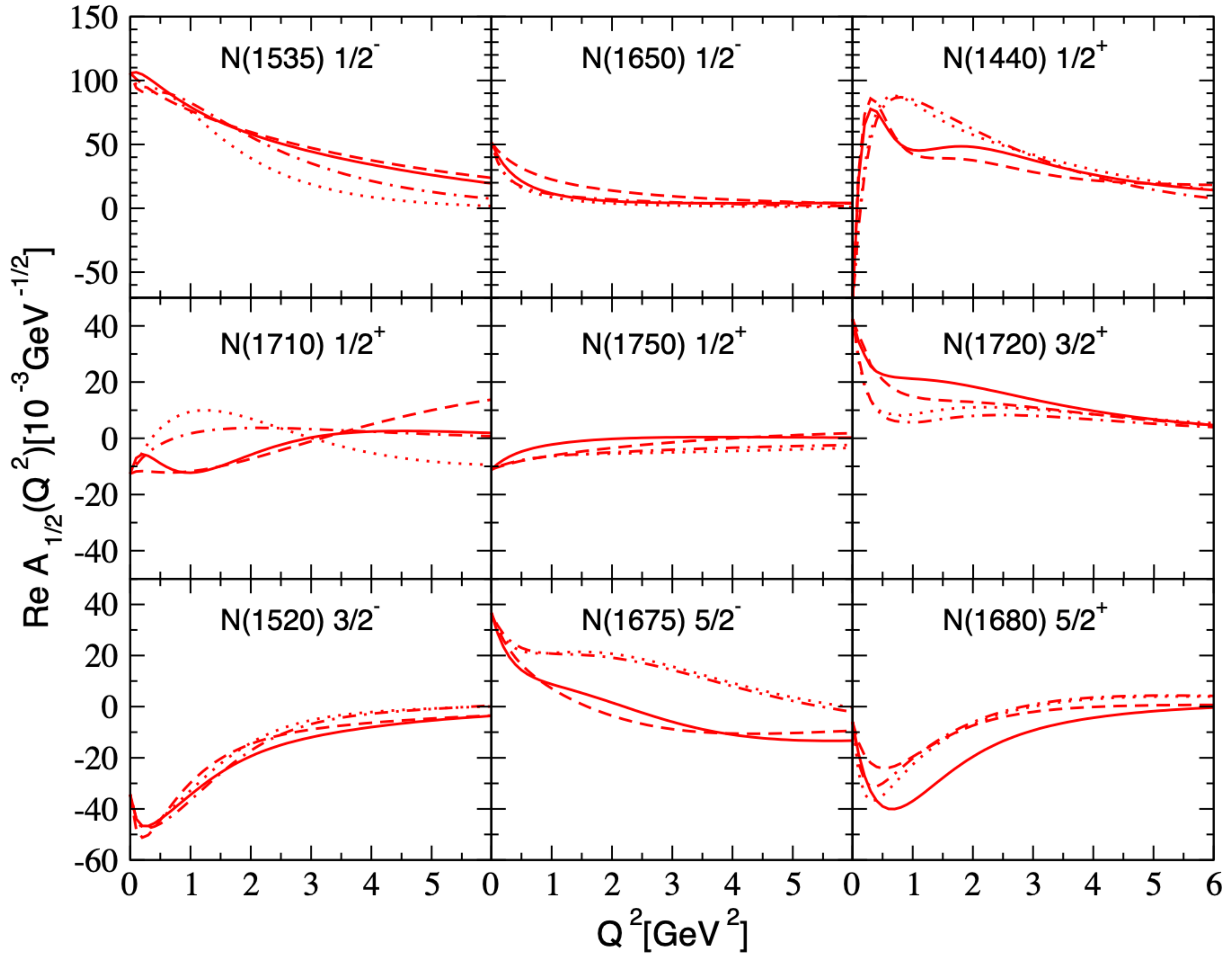
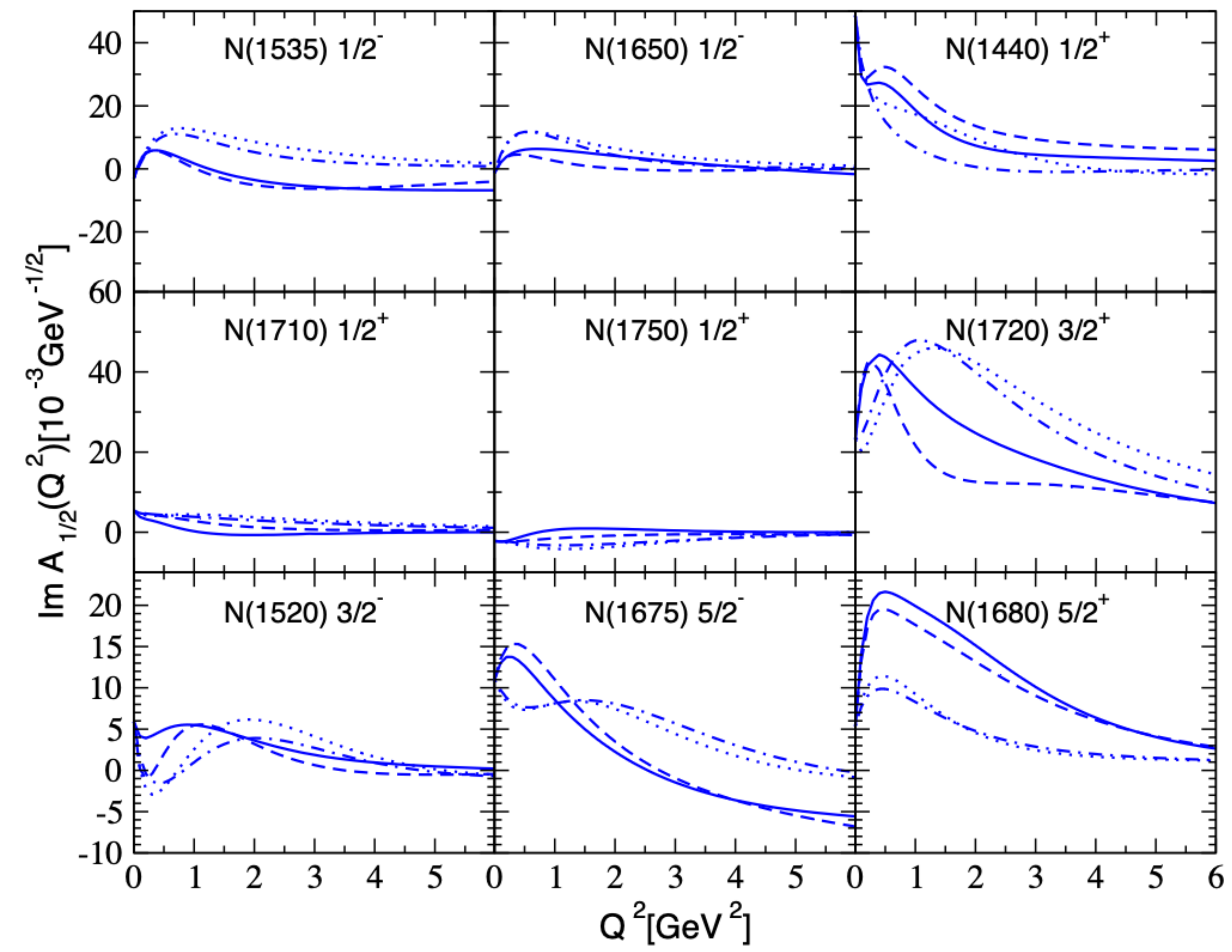


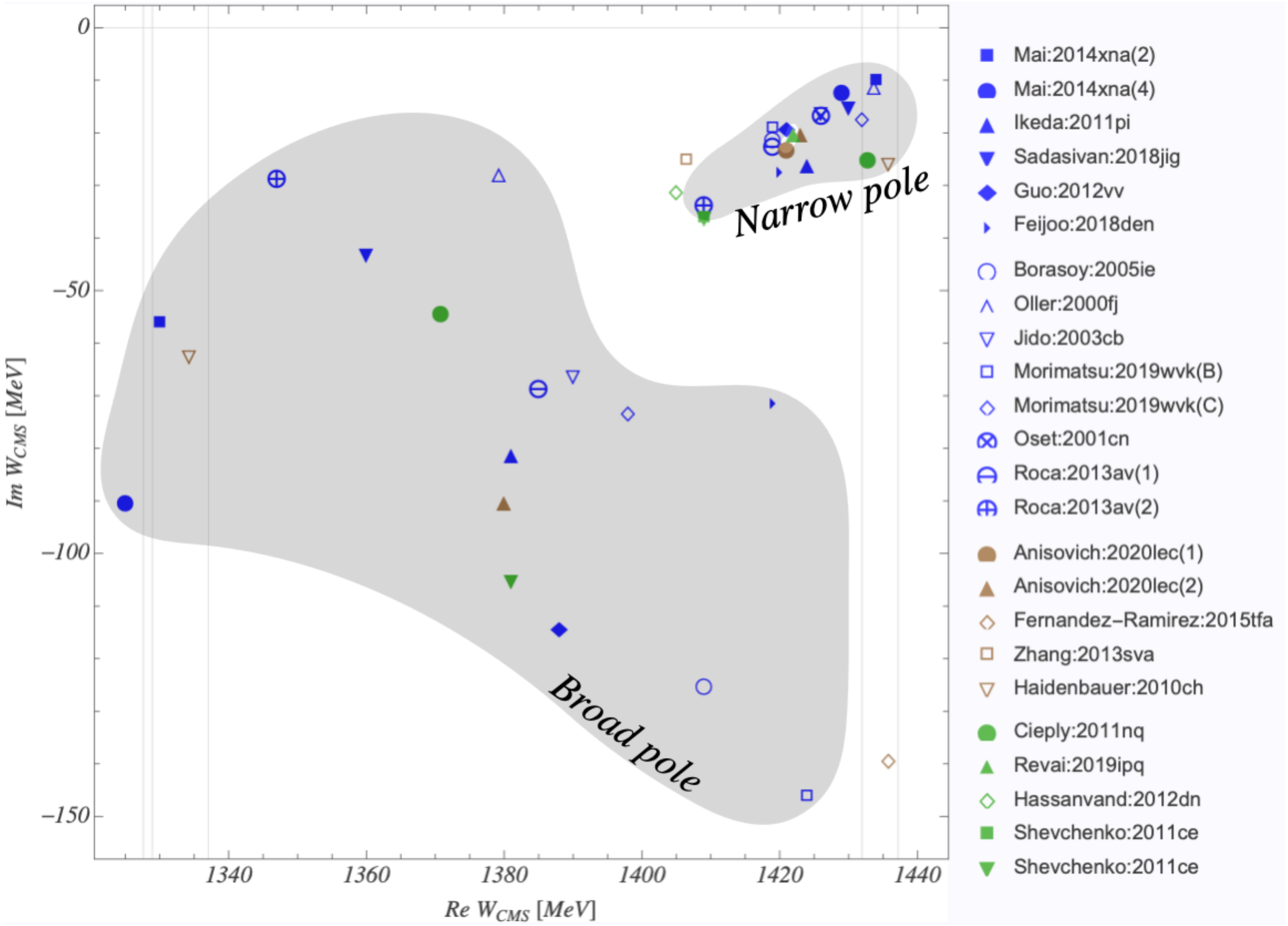
FIG. 1: The TFFs of $\Delta(1232)$. Solid, dashed, dotted, dash-dotted lines: results of this work, corresponding to fits 1 to 4 in Ref. [34]. Double-dotted lines: results from Ref. [30] based on the MAID results. Black circles: MAID results from the unitary isobar model (real-valued). Triangles: preliminary results of the ANL-Osaka model.



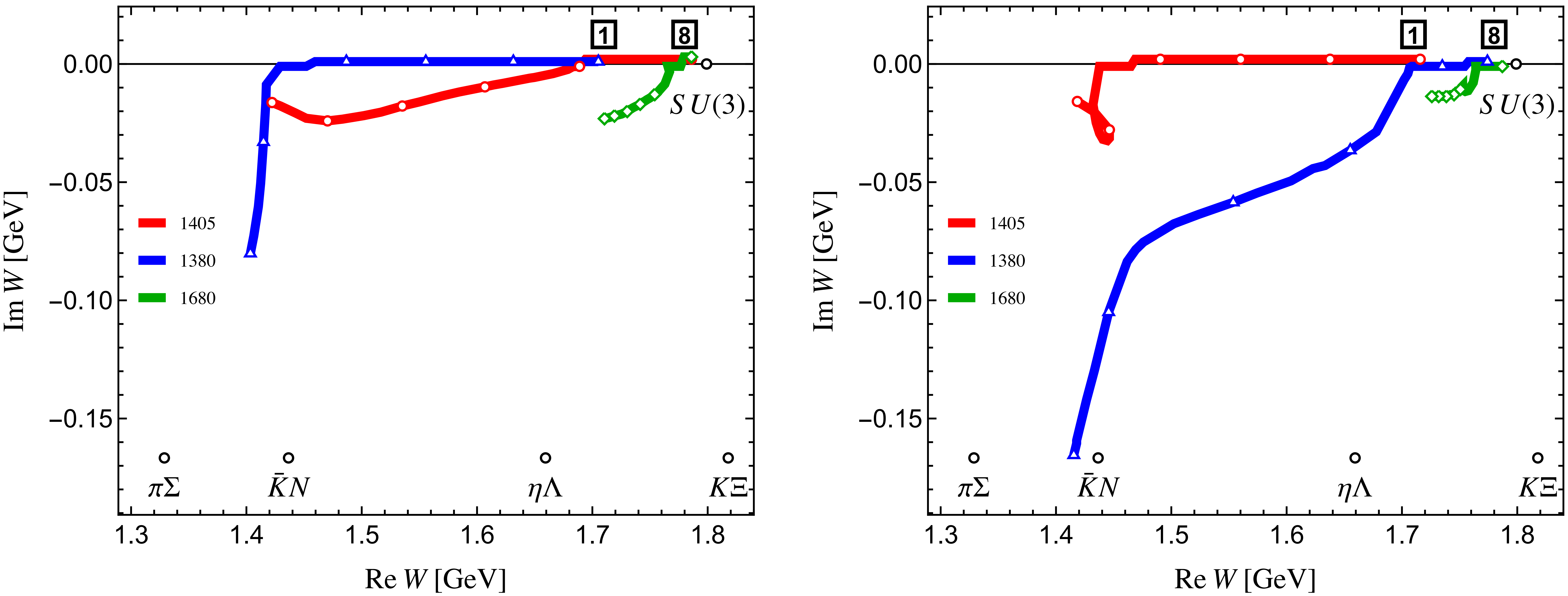


…

- Theory frontier: NNLO UCHPT determination[1]
- Consistently two poles, but the second pole is less well known
  - second pole below KbarN threshold
  - line-shape only through $\gamma p \to K\pi\Sigma$ [2]

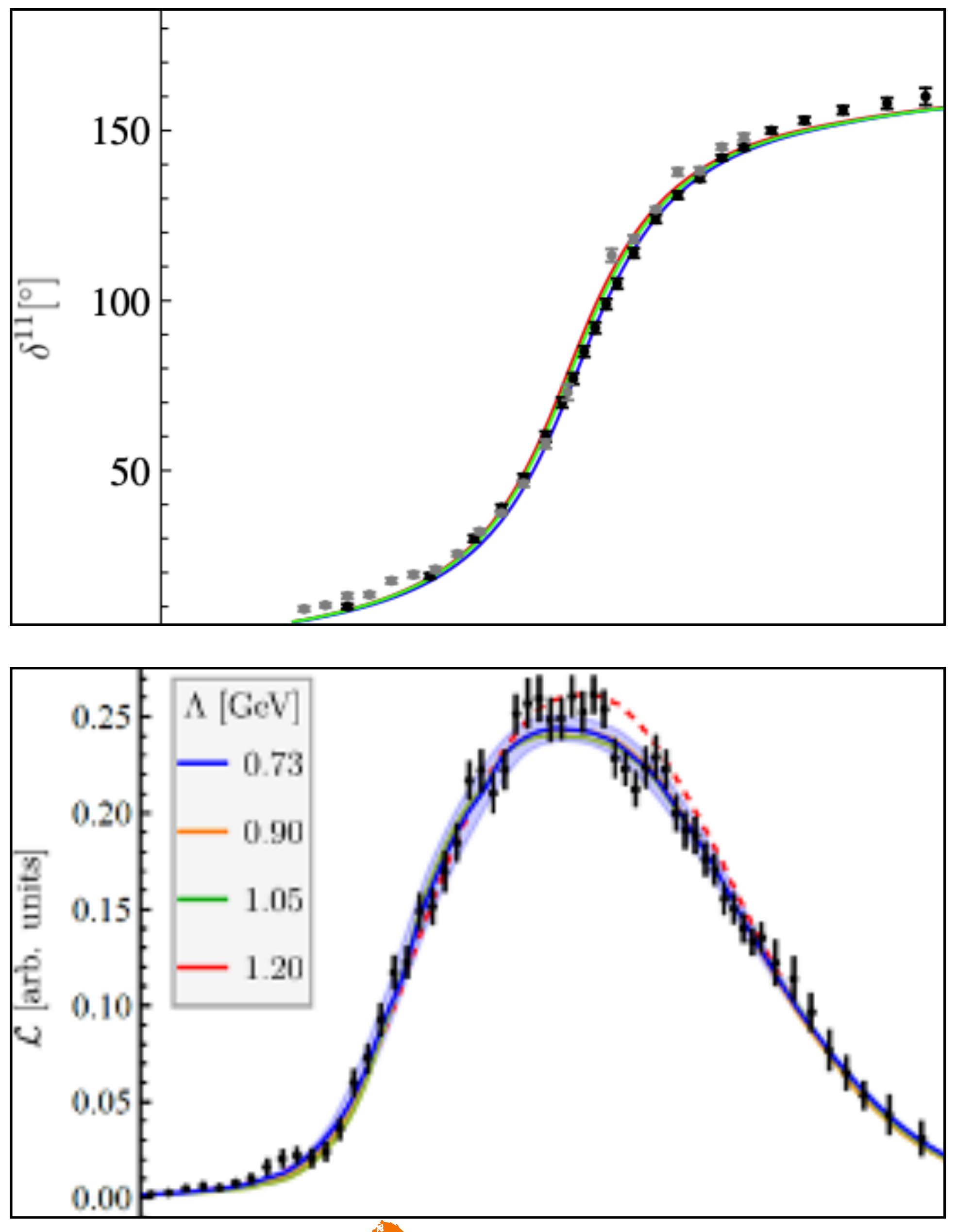


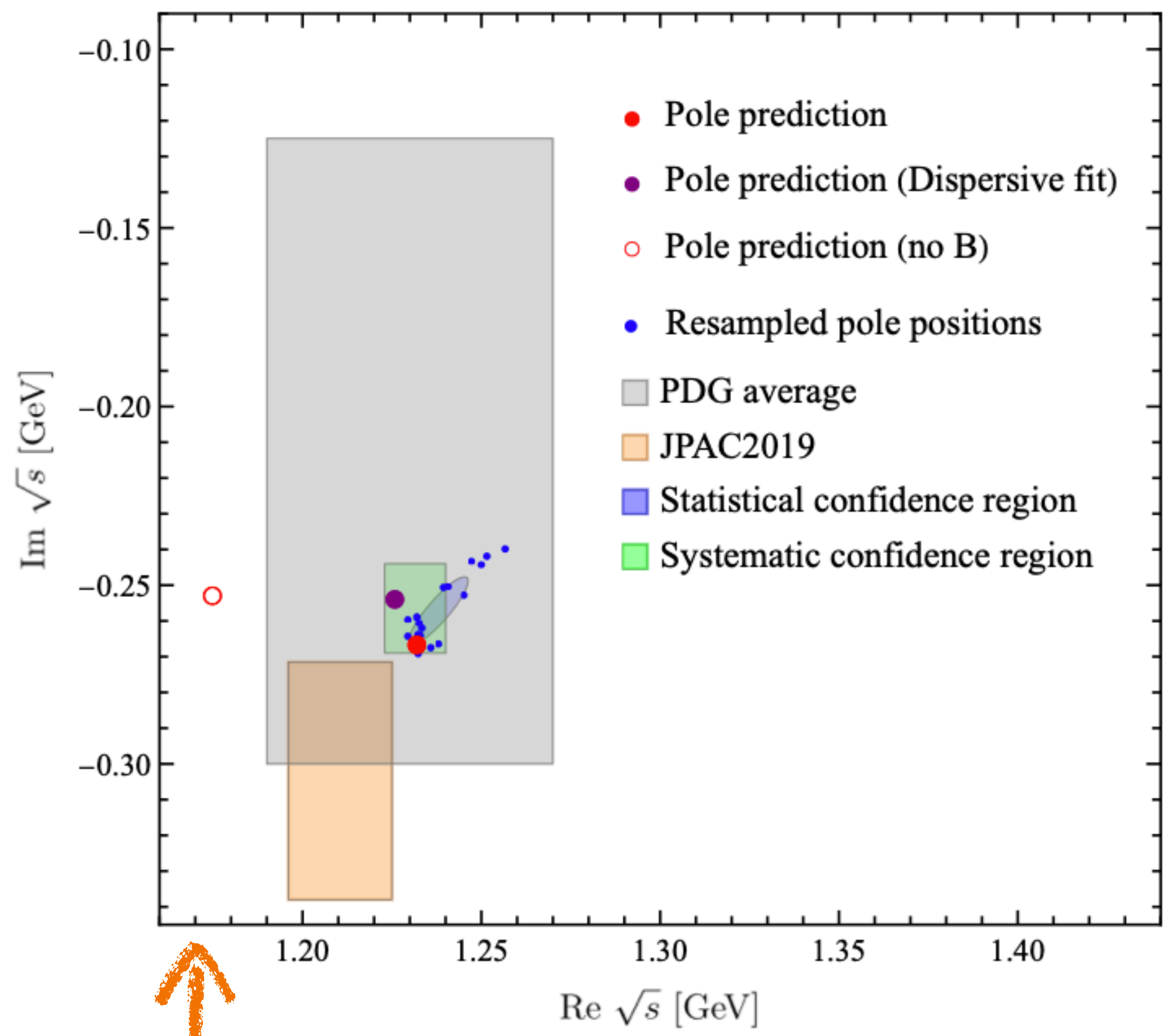
[1] Lu/Geng/Döring/MM Phys.Rev.Lett. 130 (2023)
[2] [CLAS] Moriya et al (2013)



1
8
0.00
−0.05
−0.10
−0.15
Im W [GeV]
1405
1380
1680
SU(3)
πΣ
K̄N
ηΛ
KΞ
1.3
1.4
1.5
1.6
1.7
1.8
Re W [GeV]

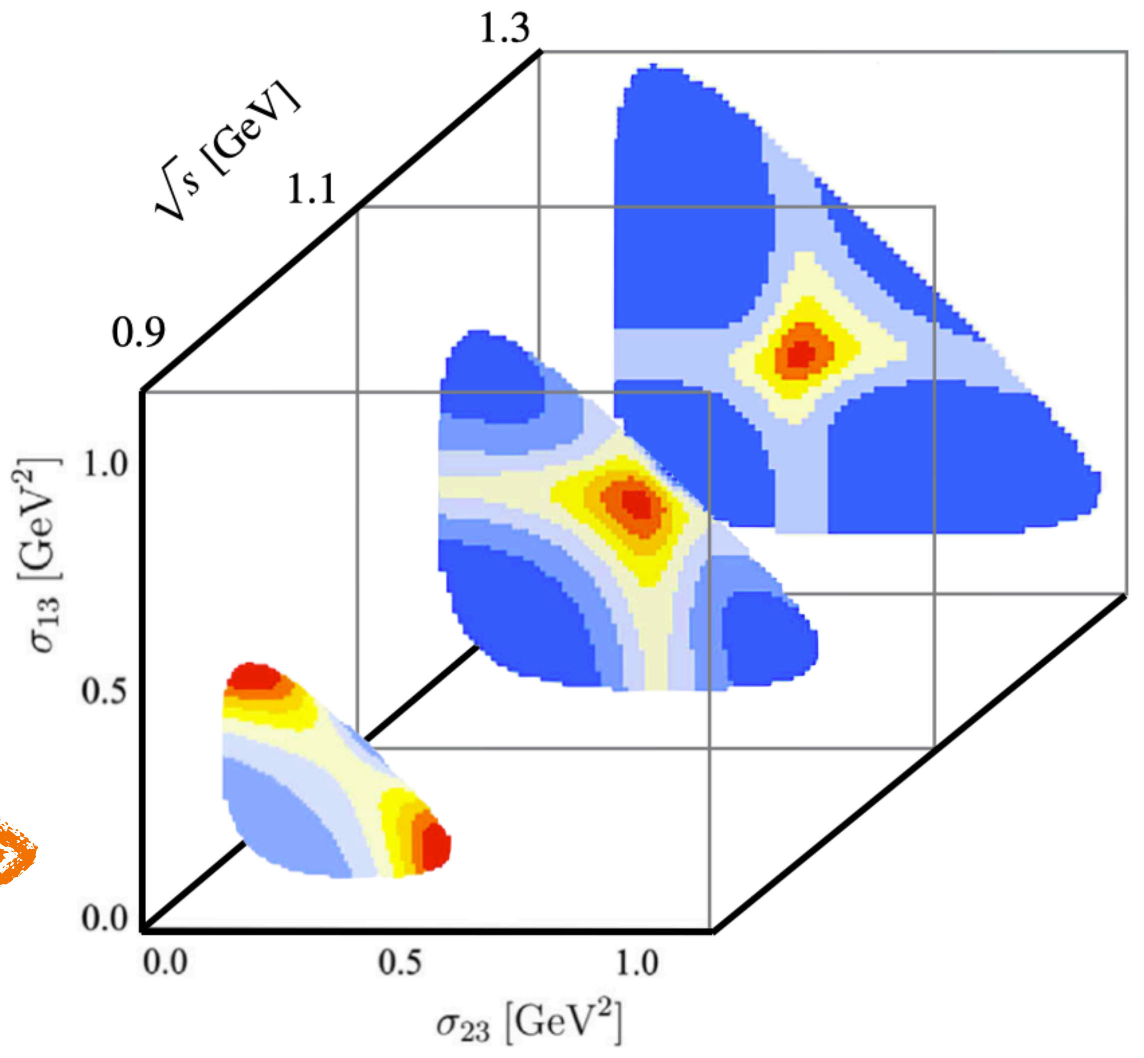
APPLICATION: $a_1(1260)$





- $\pi\rho$ dynamics dominates the $1^-(1^{++})$ system
- Integral equation solved
  - Helicity formalism
  - complex momentum mapping
- $\pi\rho/\pi\sigma/\pi(\pi\pi)_2$ extended…

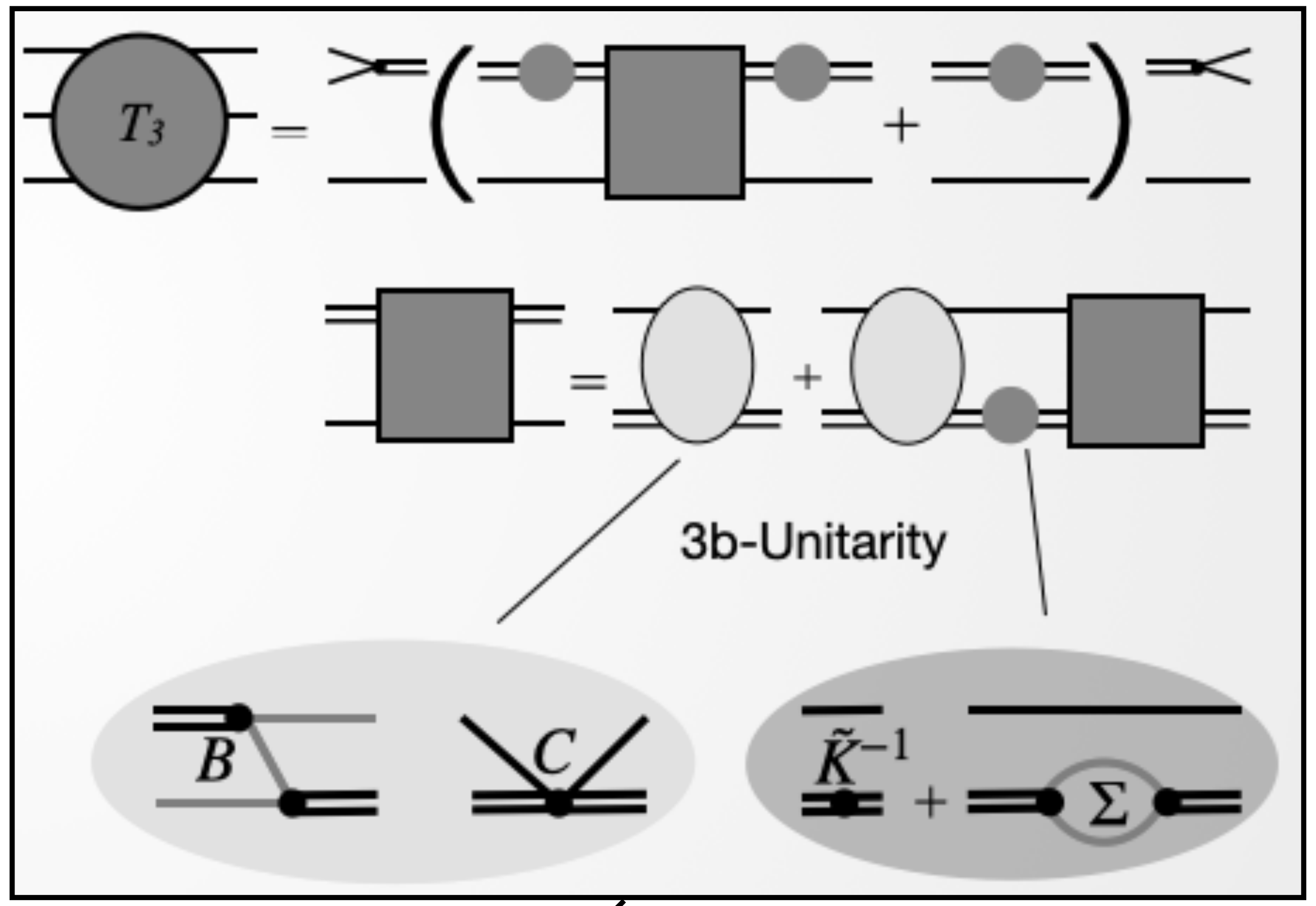
$$T^c = B + C + \int \frac{d^3\ell}{(2\pi)^3} \frac{(B+C)}{2E_l} \frac{1}{\tilde{K}_n^{-1} - \Sigma_n} T^c$$



Data: Schael [ALEPH] Phys.Rept. 421 (2005); Nucl.Phys.B 79; Phys.Rev.D 7;
Pole extraction: Sadasivan/MM/Döring/Alexandru/Culver/Lee Phys.Rev.D 101 (2020)

3-BODY QUANTIZATION CONDITION (FVU)

reminder



- **Finite-volume unitarity (FVU[1])**
  - heavily simplified:
    - on-shell particle-configurations: **Δ***E*~*mL*
    - off-shell particle-configurations: **Δ***E*~*e*$^{-mL}$
  - *<u>Unitary</u> 3-body amplitude separates these effects*
  - unknown volume independent quantities (K, C)

$$0 = \det\left[2L^3E\left(\tilde{K}_n^{-1}-\Sigma\right)-B-C\right]_{\mathbf{p}'\mathbf{p}}$$

[1] MM/Döring Phys.Rev.Lett. 122 (2019) 6

Reviews: Hansen/Sharpe Ann.Rev.Nucl.Part.Sci. 69 (2019); MM/Doring/Rusetsky Eur.Phys.J.ST 230 (2021);

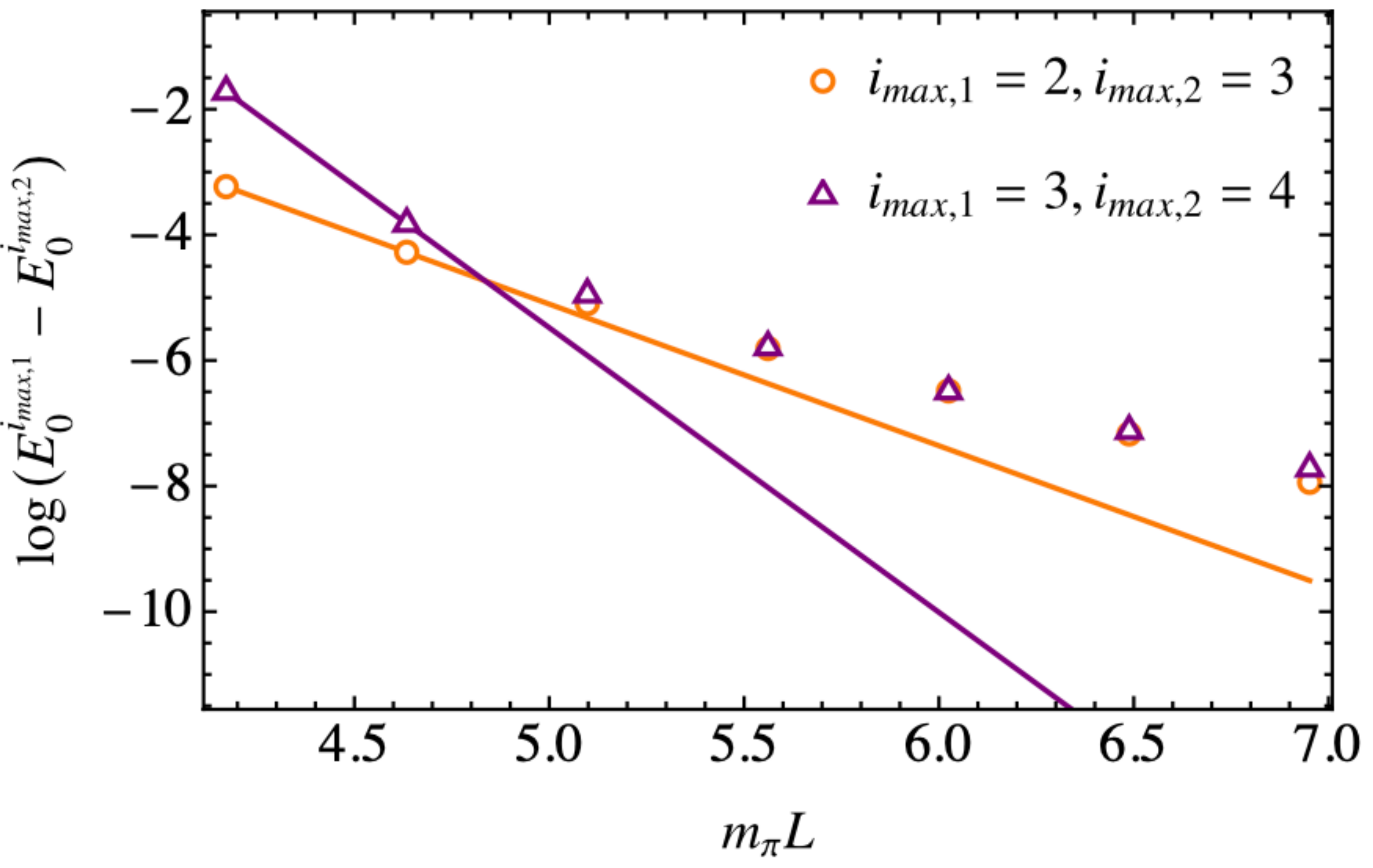
CUTOFF DEPENDENCE[1]

Consider fixed C, K then increase hard cutoff

- 3-body amplitude = genuine integral equation
  - spectator can carry arbitrary momentum away
  - cutoff required (form factors, hard cutoff,…)

$$0 = \det\left[2L^3E\left(\tilde{K}_n^{-1} - \Sigma\right) - B - C\right]_{\mathbf{p}'\mathbf{p}}$$

$$B(\sqrt{s}) = \frac{1}{\sqrt{s} - \sqrt{s_{\text{on}}} + i\epsilon}$$



- energy eigenvalues change slower than $\Delta E \sim e^{-mL}$
- one-particle exchange falls off not rapidly enough

[1] paper in preparation

CUTOFF DEPENDENCE[1]

Consider fixed C, K then increase <u>hard</u> cutoff

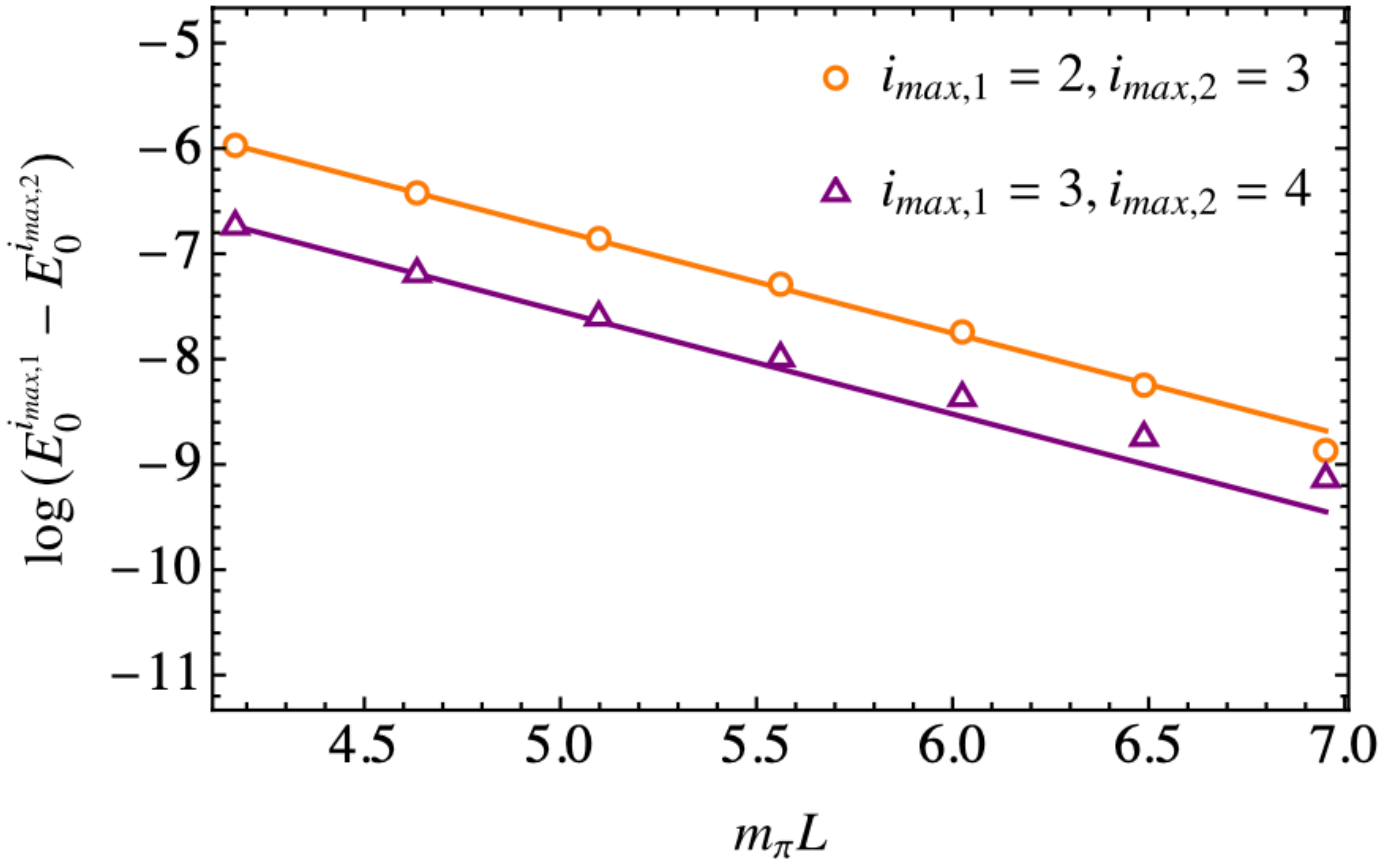
… over-subtract OPE

$$\overline{B}(\sqrt{s}) = B(0) + B'(0)\sqrt{s} + \frac{s}{s_{\mathrm{on}}}\frac{N}{2E_{p+p'}}\frac{1}{\sqrt{s}-\sqrt{s_{\mathrm{on}}}+i\epsilon}$$

- 3-body amplitude = genuine integral equation
  - spectator can carry arbitrary momentum away
  - cutoff required (form factors, hard cutoff,…)

$$0 = \det\left[2L^3E\left(\tilde{K}_n^{-1} - \Sigma\right) - B - C\right]_{\mathbf{p}'\mathbf{p}}$$

$$B(\sqrt{s}) = \frac{1}{\sqrt{s}-\sqrt{s_{\mathrm{on}}}+i\epsilon}$$



- energy eigenvalues change as $\Delta E \sim e^{-mL}$

[1] paper in preparation

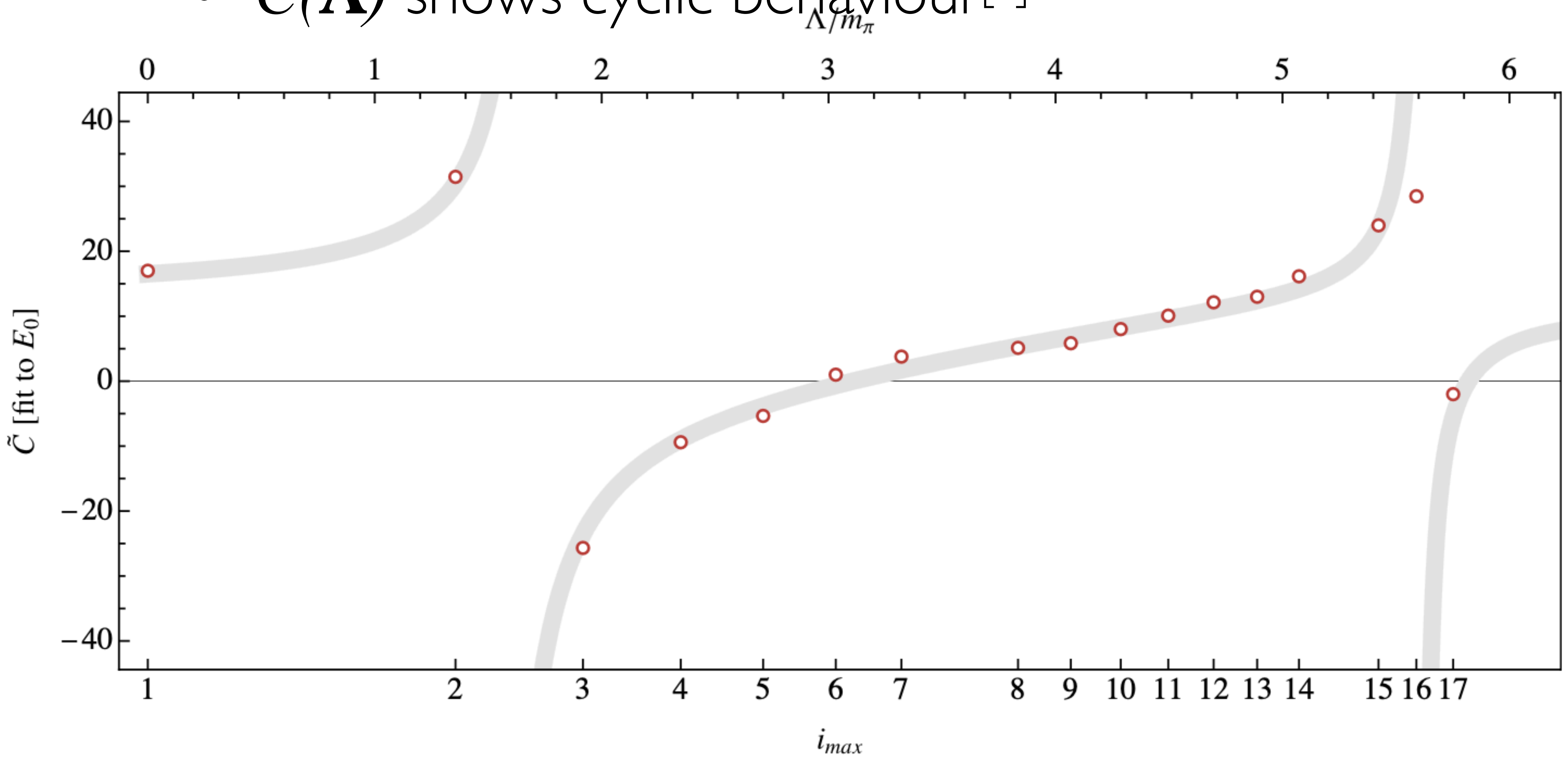
CUTOFF DEPENDENCE[1]

- 3-body amplitude = genuine integral equation
  - spectator can carry arbitrary momentum away
  - cutoff required (form factors, hard cutoff,…)

$$0 = \det\left[2L^3E\left(\tilde{K}_n^{-1} - \Sigma\right) - B - C\right]_{\mathbf{p}'\mathbf{p}}$$

Consider fixed ground-state finite-volume level (E_0)

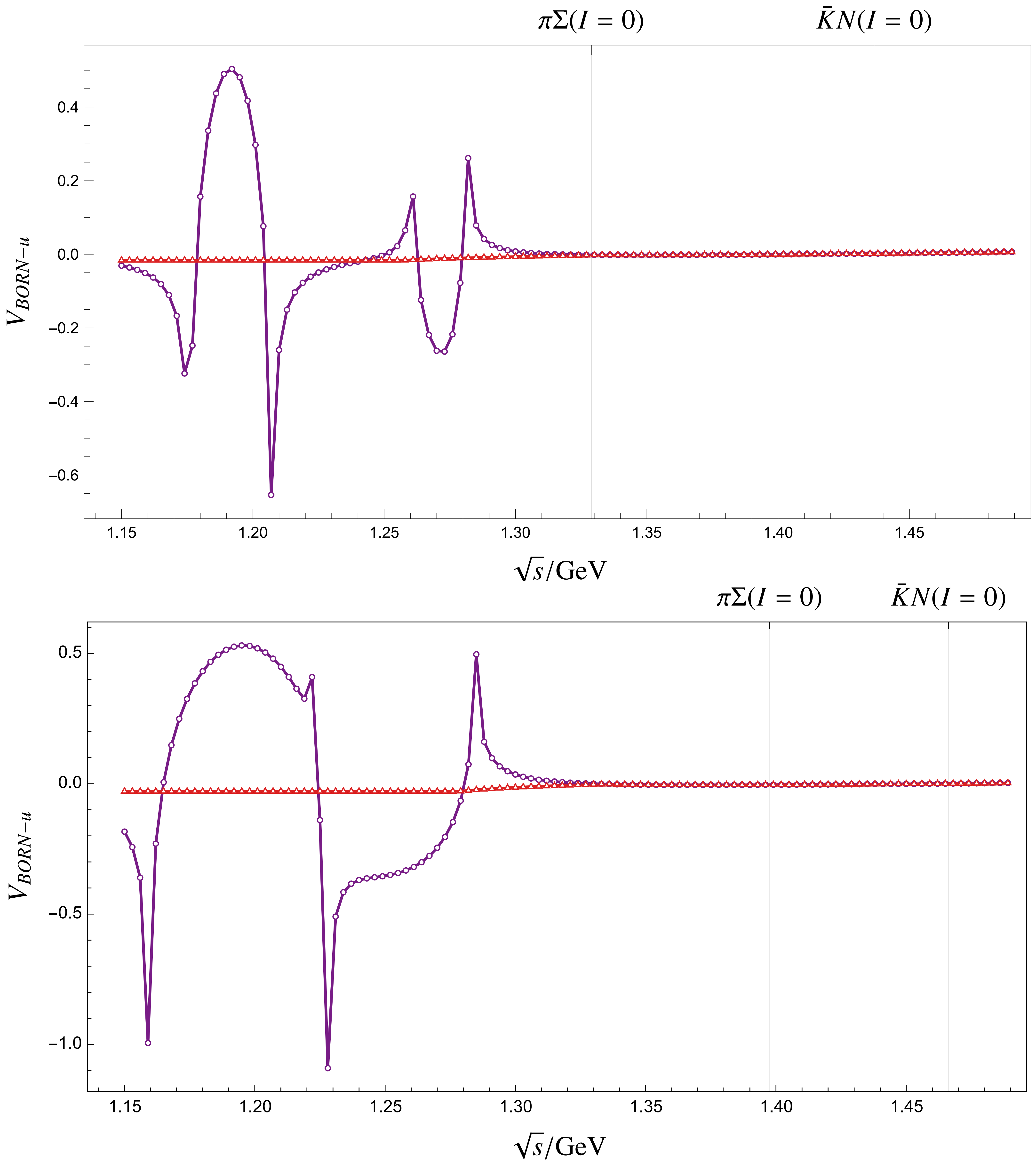
- change cutoff & refit C
- $\pi\rho/\pi(\pi\pi)_2$ repulsiv system
- $C(\Lambda)$ shows cyclic behaviour[2]



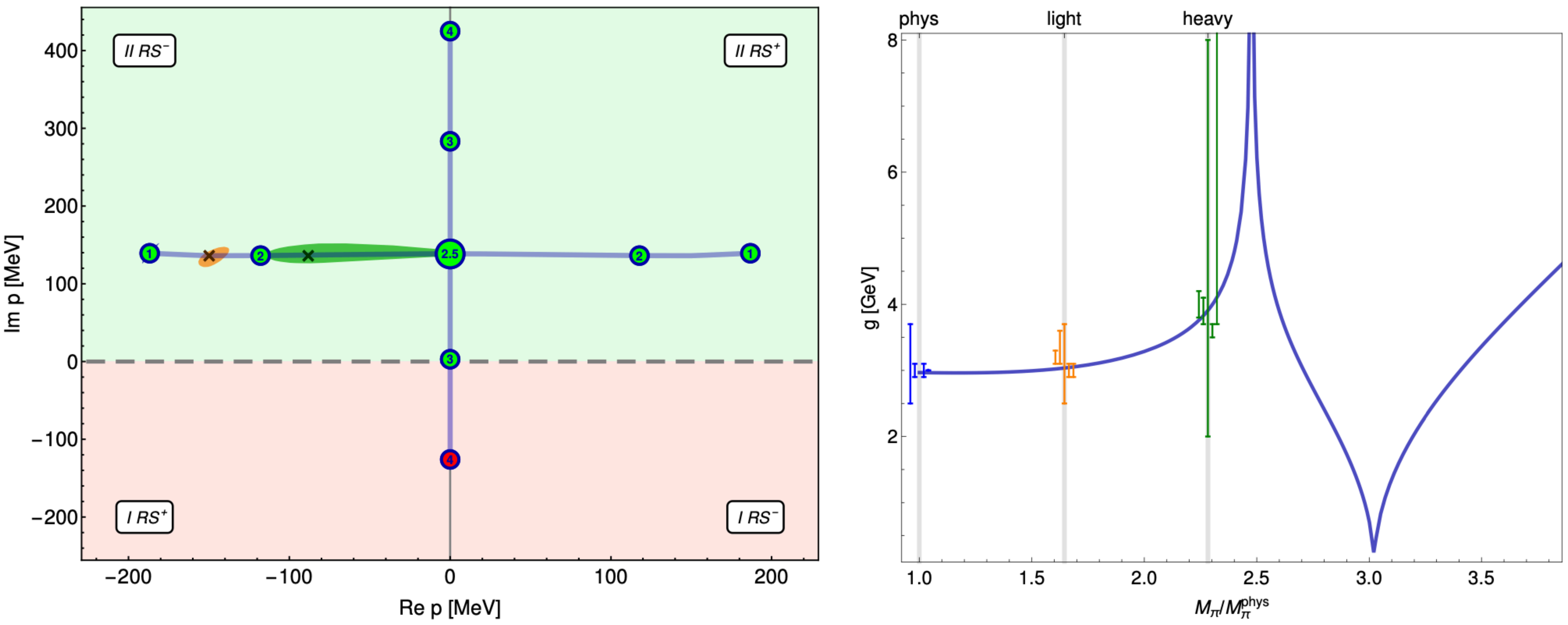
[1] paper in preparation
[2] Bedaque/Hammer/van Kolck, Phys. Rev. Lett. 82 (1999) 463; Bedaque/Hammer/van Kolck, Nucl.Phys. A 646 (1999) 444

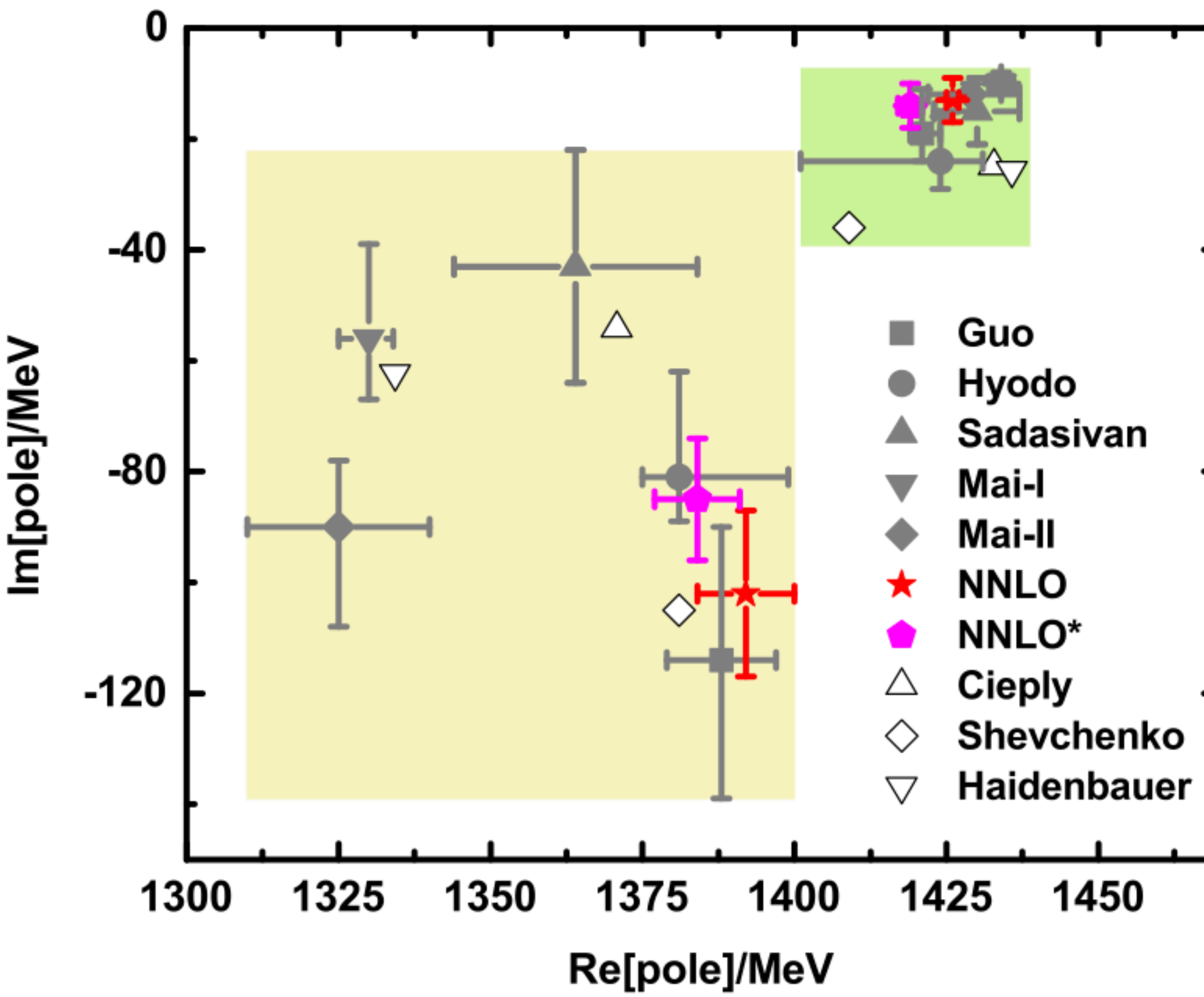
U-CHANNEL IN THE Λ(1405)

- New insights[1] from LQCD [next talk]
  - confirming two-pole scenario
- Chiral extrapolations (through UCHPT)[2]
  - u-channel baryon exchange may complicate the picture (3-body)
  - sub-leading effect



[1] [BaSc] Bulava et al. 2307.10413; 2307.13471
[2] Guo/Kamyia/MM/Meißner Phys.Lett.B 846 (2023)


II RS⁻
II RS⁺
I RS⁺
I RS⁻
Im p [MeV]
Re p [MeV]
400
300
200
100
0
−100
−200
200
1
2
2.5
3
4
phys
light
heavy
g [GeV]
8
6
4
2
1.0
1.5
2.0
2.5
3.0
3.5
M_π/M_π^{phys}



0
-40
-80
-120
1300
1325
1350
1375
1400
1425
1450
Im[pole]/MeV
Re[pole]/MeV
Guo
Hyodo
Sadasivan
Mai-I
Mai-II
NNLO
NNLO*
Cieply
Shevchenko
Haidenbauer

$$\{1, 8_s, 8_a, 10, \overline{10}, 27\}$$

$$\begin{pmatrix} |\pi\Sigma\rangle \\ |\bar{K}N\rangle \\ |\eta\Lambda\rangle \\ |K\Xi\rangle \end{pmatrix} = \frac{1}{\sqrt{40}} \begin{pmatrix} \sqrt{15} & -\sqrt{24} & 0 & -1 \\ -\sqrt{10} & -2 & \sqrt{20} & -\sqrt{6} \\ -\sqrt{5} & -\sqrt{8} & 0 & 3\sqrt{3} \\ \sqrt{10} & 2 & 2\sqrt{5} & \sqrt{6} \end{pmatrix} \begin{pmatrix} |1\rangle \\ |8\rangle \\ |8'\rangle \\ |27\rangle \end{pmatrix},$$

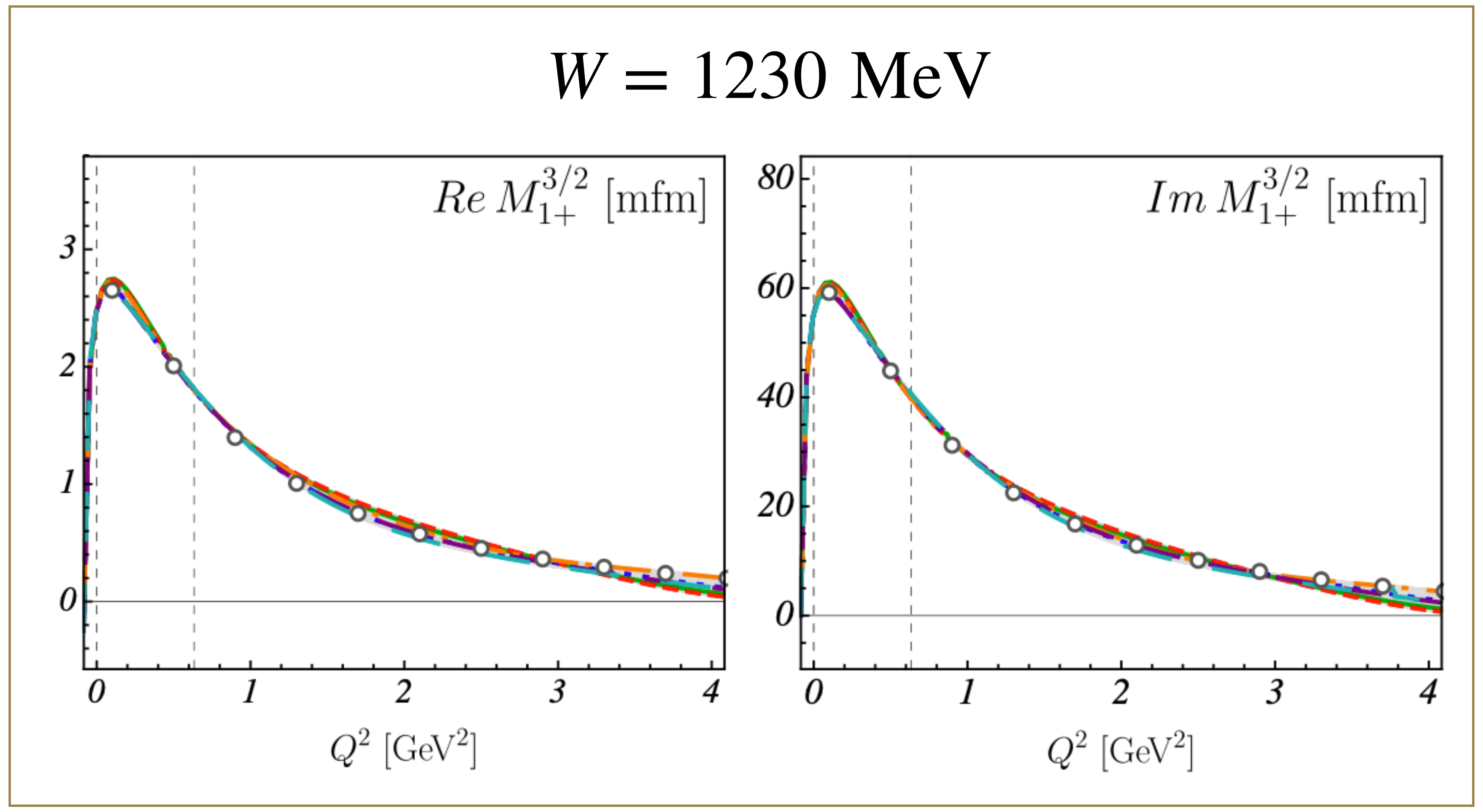
$$C_{\alpha\beta} = \begin{pmatrix} 6 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \quad \text{for} \quad \alpha, \beta \in \{1, 8, 8', 27\}.$$

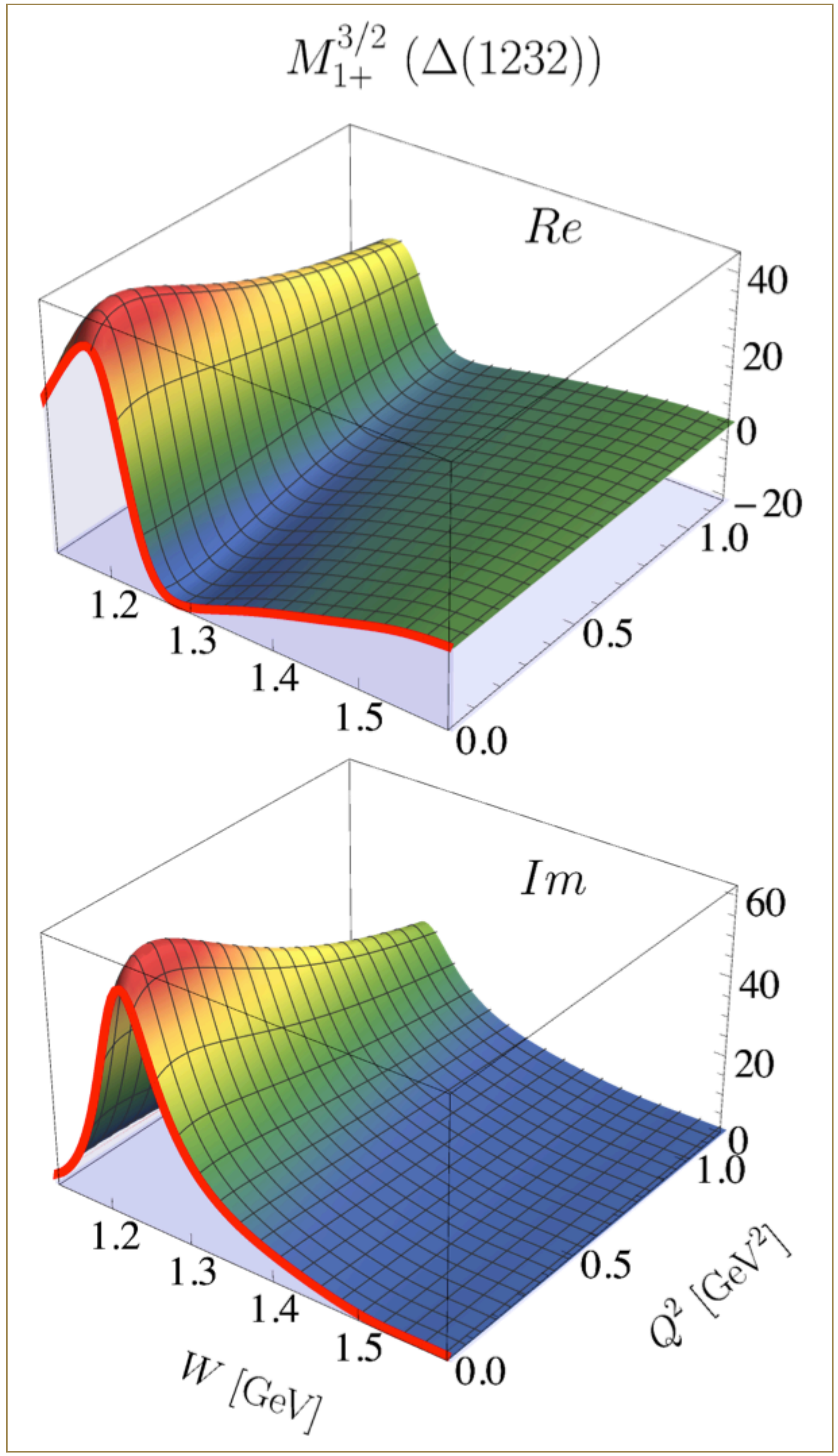
$$C^{\text{NLO1}}_{\alpha\beta} = \begin{pmatrix} \frac{4}{3}(3b_0 + 7b_D)m_q & 0 & 0 & 0 \\ 0 & \frac{2}{3}(6b_0 + b_D)m_q & -\sqrt{20}b_F m_q & 0 \\ 0 & -\sqrt{20}b_F m_q & 2(2b_0 + 3b_D)m_q & 0 \\ 0 & 0 & 0 & 4(b_0 + b_D)m_q \end{pmatrix},$$

$$C^{\text{NLO2}}_{\alpha\beta} = \begin{pmatrix} -3d_2 + \frac{9}{2}d_3 + d_4 & 0 & 0 & 0 \\ 0 & \frac{1}{2}(-3d_2 + d_3 + 2d_4) & -\frac{\sqrt{5}}{2}d_1 & 0 \\ 0 & -\frac{\sqrt{5}}{2}d_1 & \frac{1}{2}(9d_2 - d_3 + 2d_4) & 0 \\ 0 & 0 & 0 & \frac{1}{2}(2d_2 + d_3 + 2d_4) \end{pmatrix}.$$

NLO breaks accidental octet symmetry

RESULTS





Delta(1232):

- Large multipoles well determined
- simple Q^2 dependence

[JBW] MM et al. *Phys.Rev.C* 103 (2021) 6; Phys.Rev.C 106 (2022) 015201

HADRONIC 3-BODY PROBLEM: IMPACT

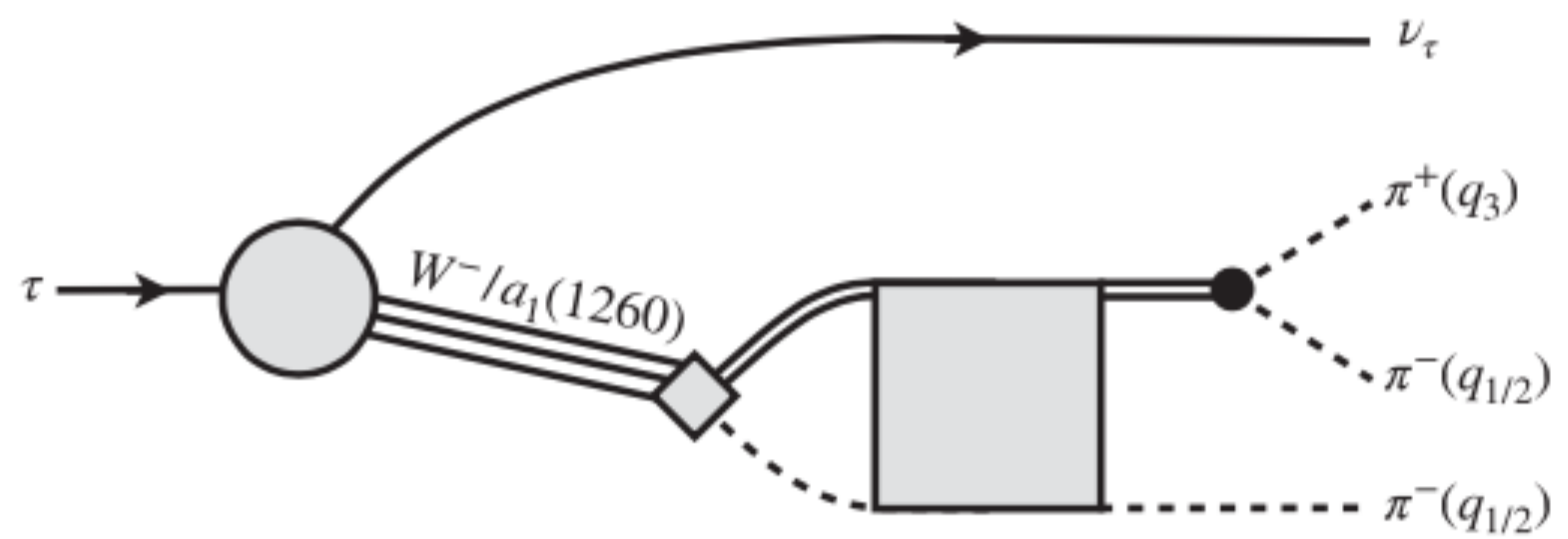
Intricate kinematics/dynamics

- 8 variables
- 2-body sub-channel dynamics

Hadron spectroscopy riddles

- Roper(1440) → **ππN** [first FV evaluations[1]]
- X(3872) → **DbarDπ**
- a_1(1260) → **πππ**
-

- Beyond Standard Model: τ-EDM



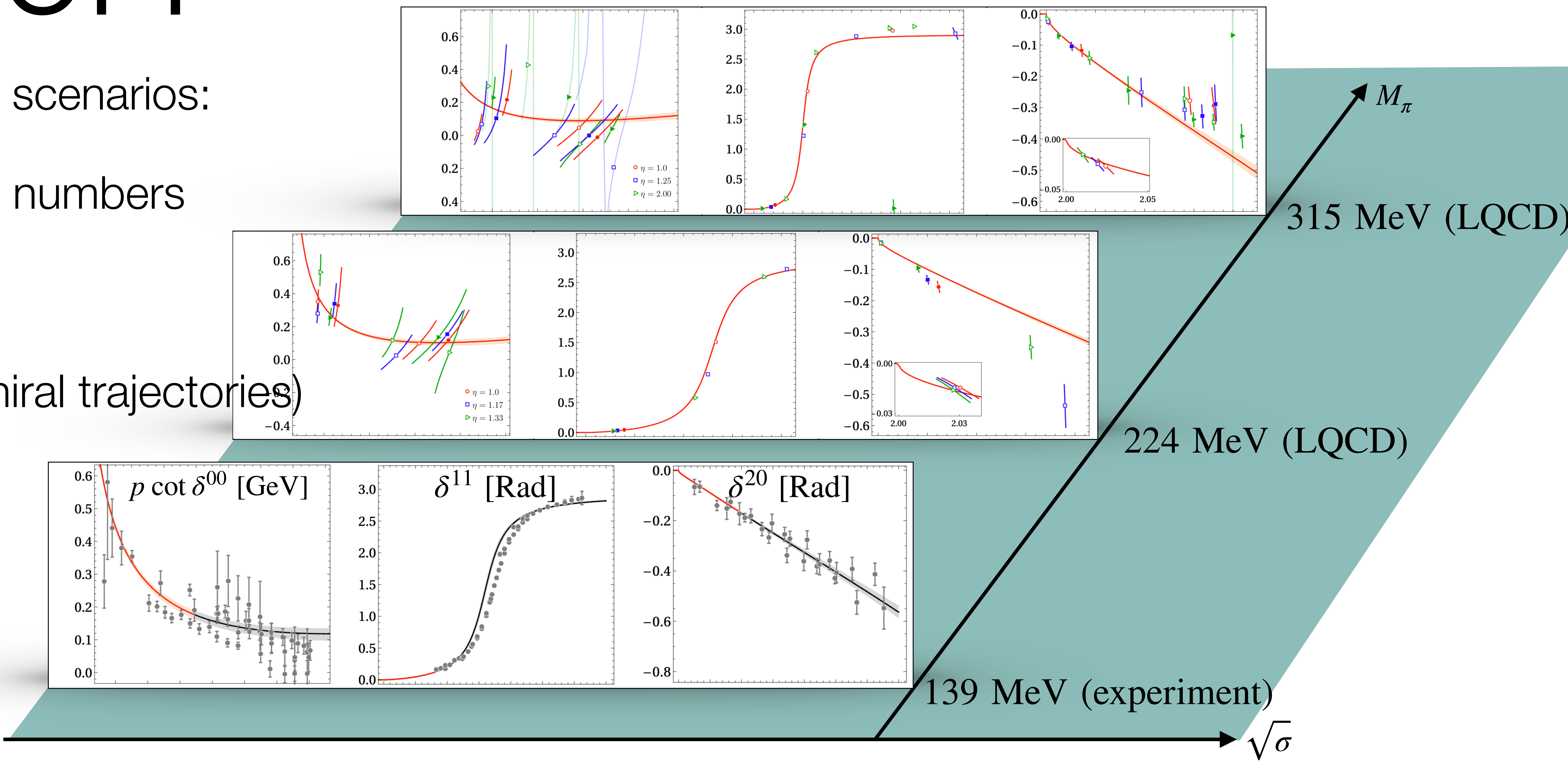
- Precision physics: rare hadronic W-decays[2]
- Exotic states of matter[3]

1) Severt/MM/Meißner JHEP04(2023) >>> PHD talk on Friday
2) Sirunyan et al. [CMS@CERN] PRL122
3) Experimental programs: GlueX@JLAB; COMPASS@CERN;

LATTICE HADRON SPECTROSCOPY

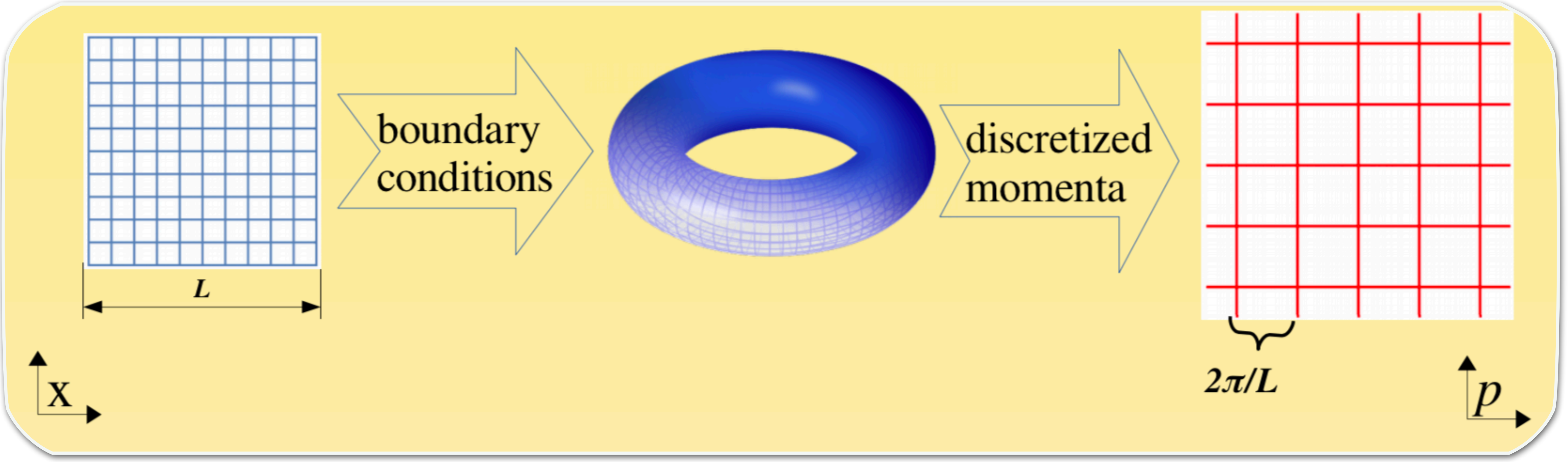
- Experimentally inaccessible scenarios:
  - Unconventional quantum numbers
  - Three-body scattering
  - Unphysical pion mass (chiral trajectories)

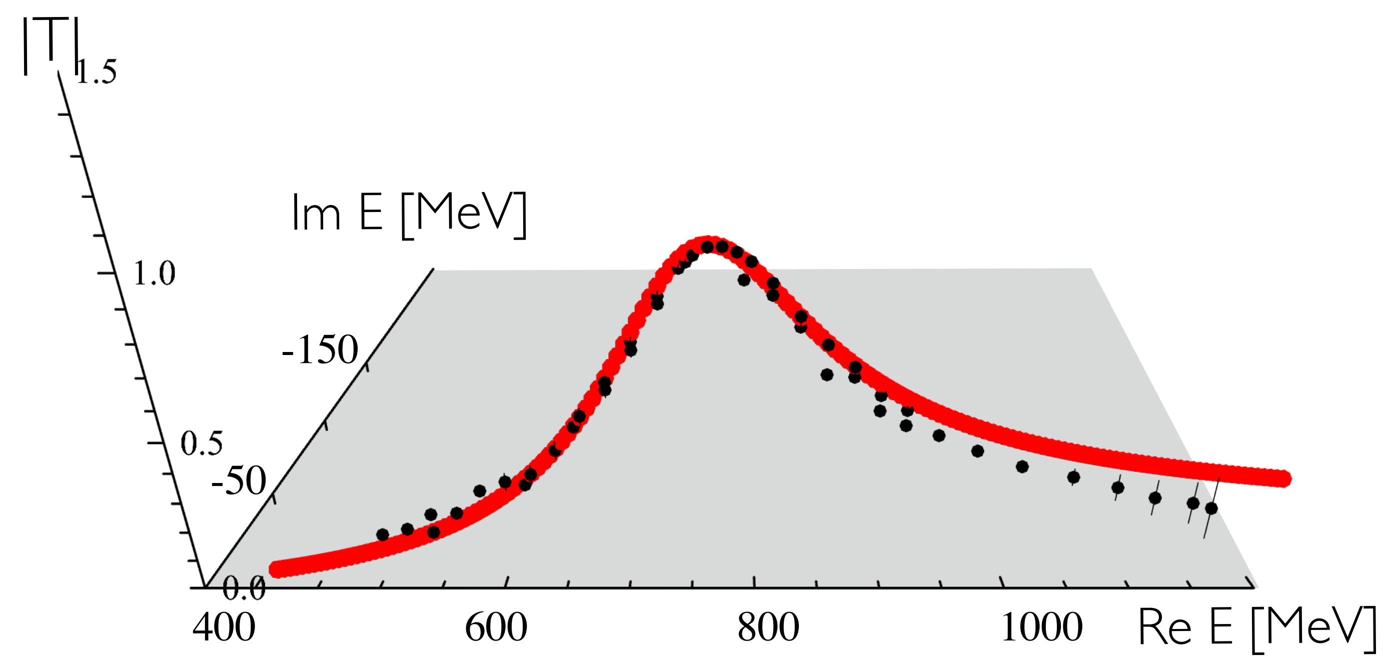
  …



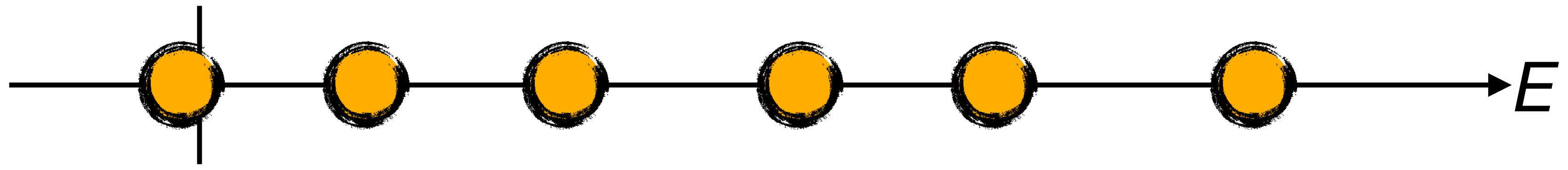
MM/Culver/Brett/Alexandru/Döring/Lee *Phys.Rev.D* 100 (2019)
Review: MM/Döring/Rusetsky EPJ ST (2021)

HADRONS IN A BOX





🤩 *A unitary "T-matrix" accounts for all O(mL) effects!*



🤗 Heavily simplified:

on-shell particle-configurations: $\boldsymbol{\Delta} E \sim mL$

off-shell particle-configurations: $\boldsymbol{\Delta} E \sim e^{-mL}$

1) Lüscher, Gottlieb, Rummukainen, Feng, Li, Döring, Briceño, Meißner, Rusetsky, Hansen, MM, Blanton, ...
2) Reviews: Hansen/Sharpe Ann.Rev.Nucl.Part.Sci. 69 (2019); MM/Döring/Rusetsky Eur.Phys.J.ST 230 (2021);

HADRONS IN A BOX

Finite-volume spectrum is real and discrete!

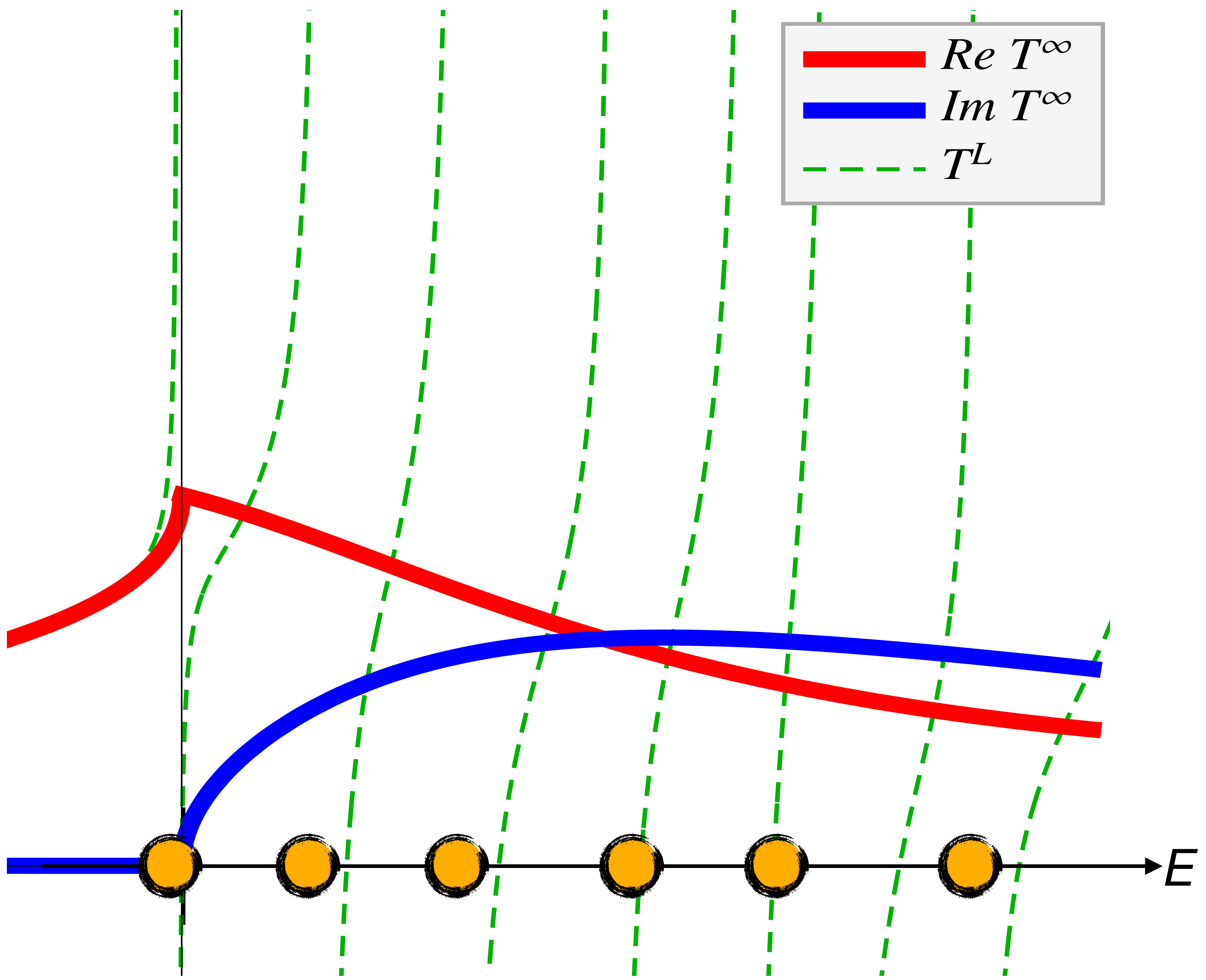
... requires mapping: Quantization condition[1,2]

🤗 Heavily simplified:

on-shell particle-configurations: $\Delta E \sim mL$

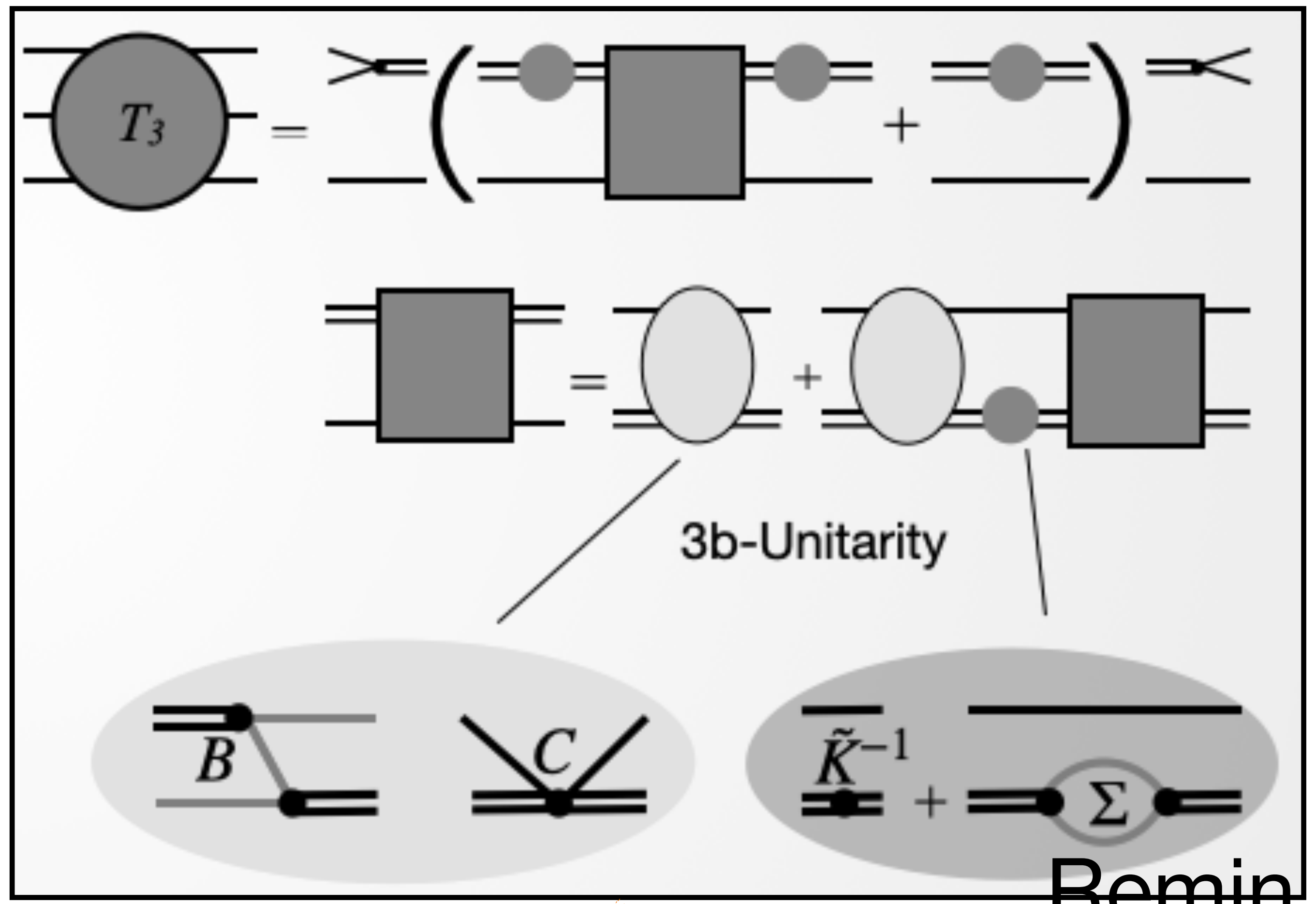
off-shell particle-configurations: $\Delta E \sim e^{-mL}$

🤩 *A unitary "T-matrix" accounts for all O(mL) effects!*



1) Lüscher, Gottlieb, Rummukainen, Feng, Li, Döring, Briceño, Meißner, Rusetsky, Hansen, MM, Blanton, ...
2) Reviews: Hansen/Sharpe Ann.Rev.Nucl.Part.Sci. 69 (2019); MM/Döring/Rusetsky Eur.Phys.J.ST 230 (2021);

3-BODY QUANTISATION CONDITION



Remin der

Finite-volume unitarity (FVU)[1,2]

- separates volume dependent terms
- volume independent terms connect infinite/finite-volume spectra

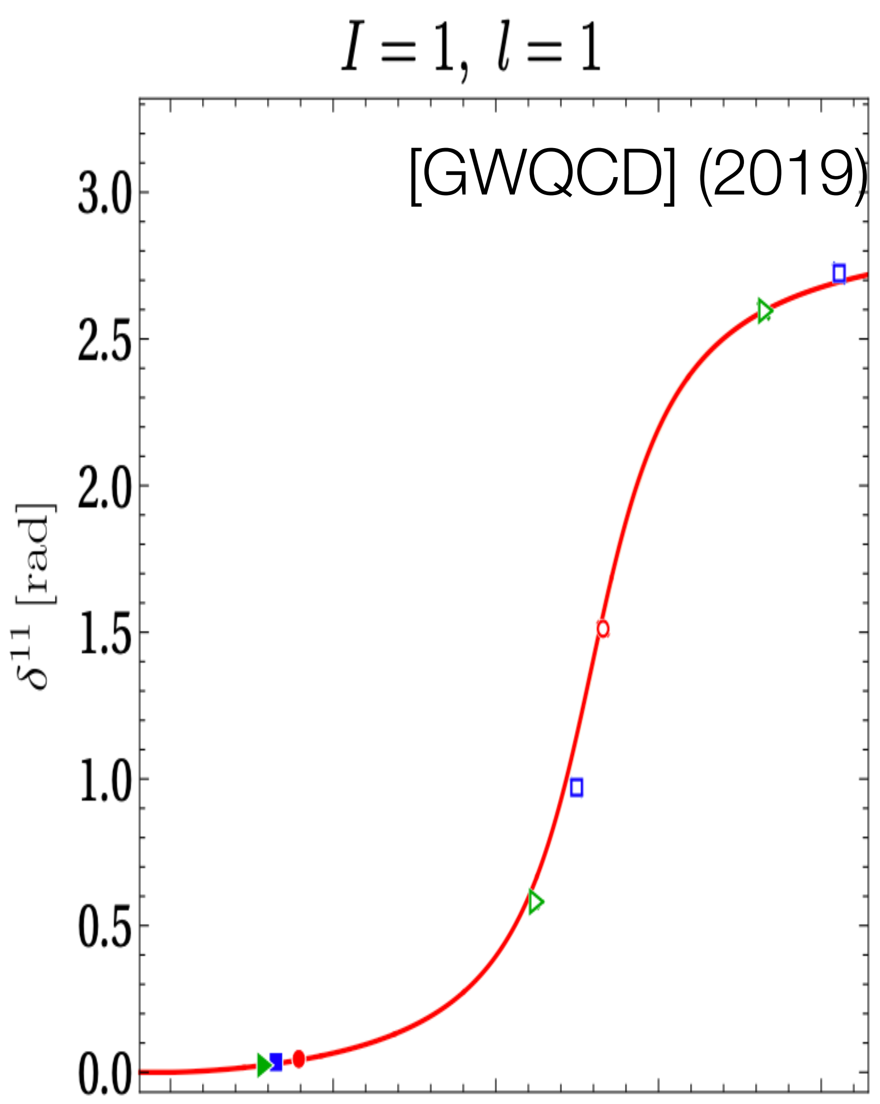
$$0 = \det\left[2L^3E\left(\tilde{K}_n^{-1} - \Sigma\right) - B - C\right]_{\mathbf{p}'\mathbf{p}}$$

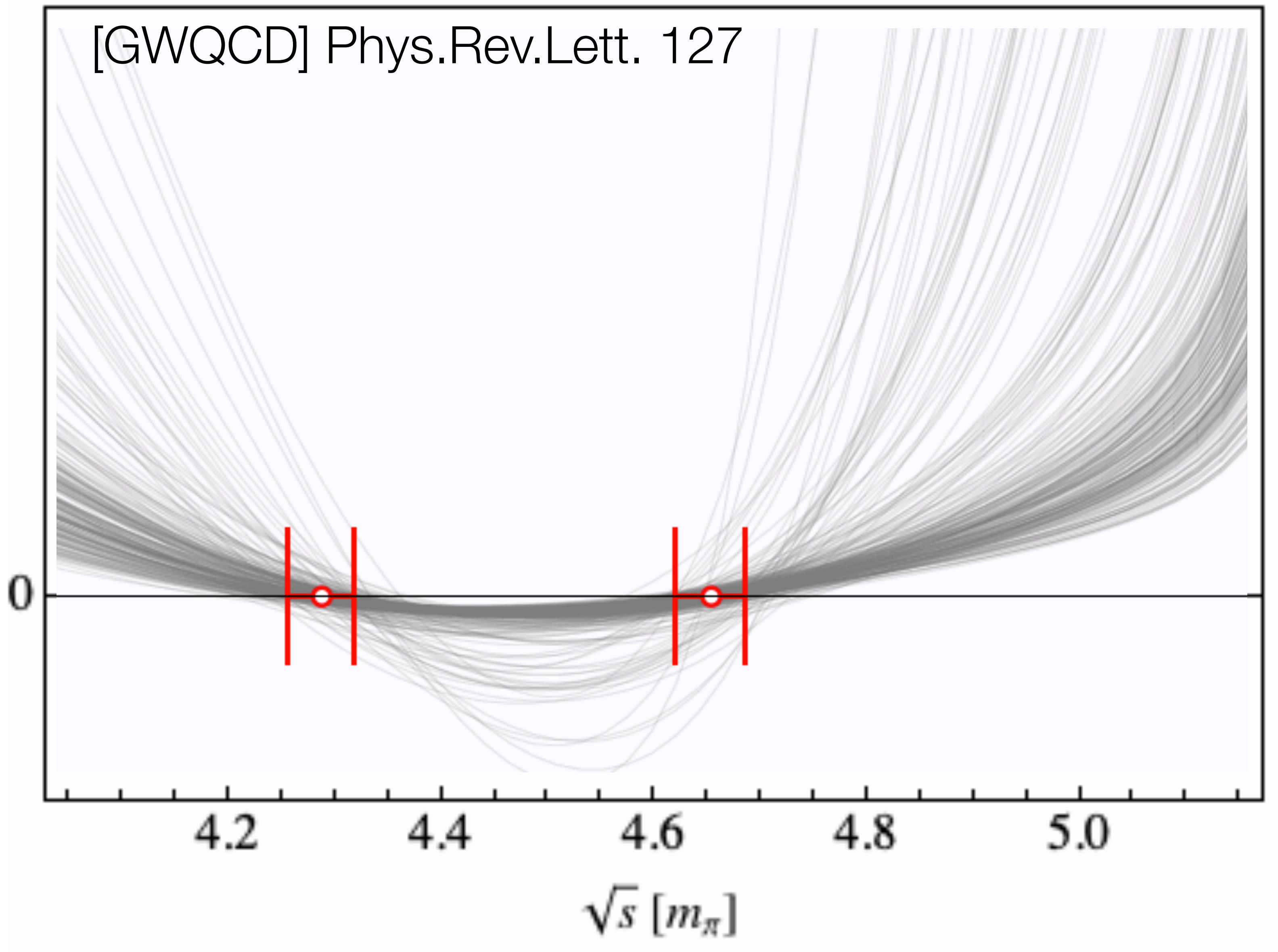
1) Lüscher, Gottlieb, Rummukainen, Feng, Li, Döring, Briceño, Meißner, Rusetsky, Hansen, MM, Blanton, ...
2) Reviews: Hansen/Sharpe Ann.Rev.Nucl.Part.Sci. 69 (2019); MM/Döring/Rusetsky Eur.Phys.J.ST 230 (2021);

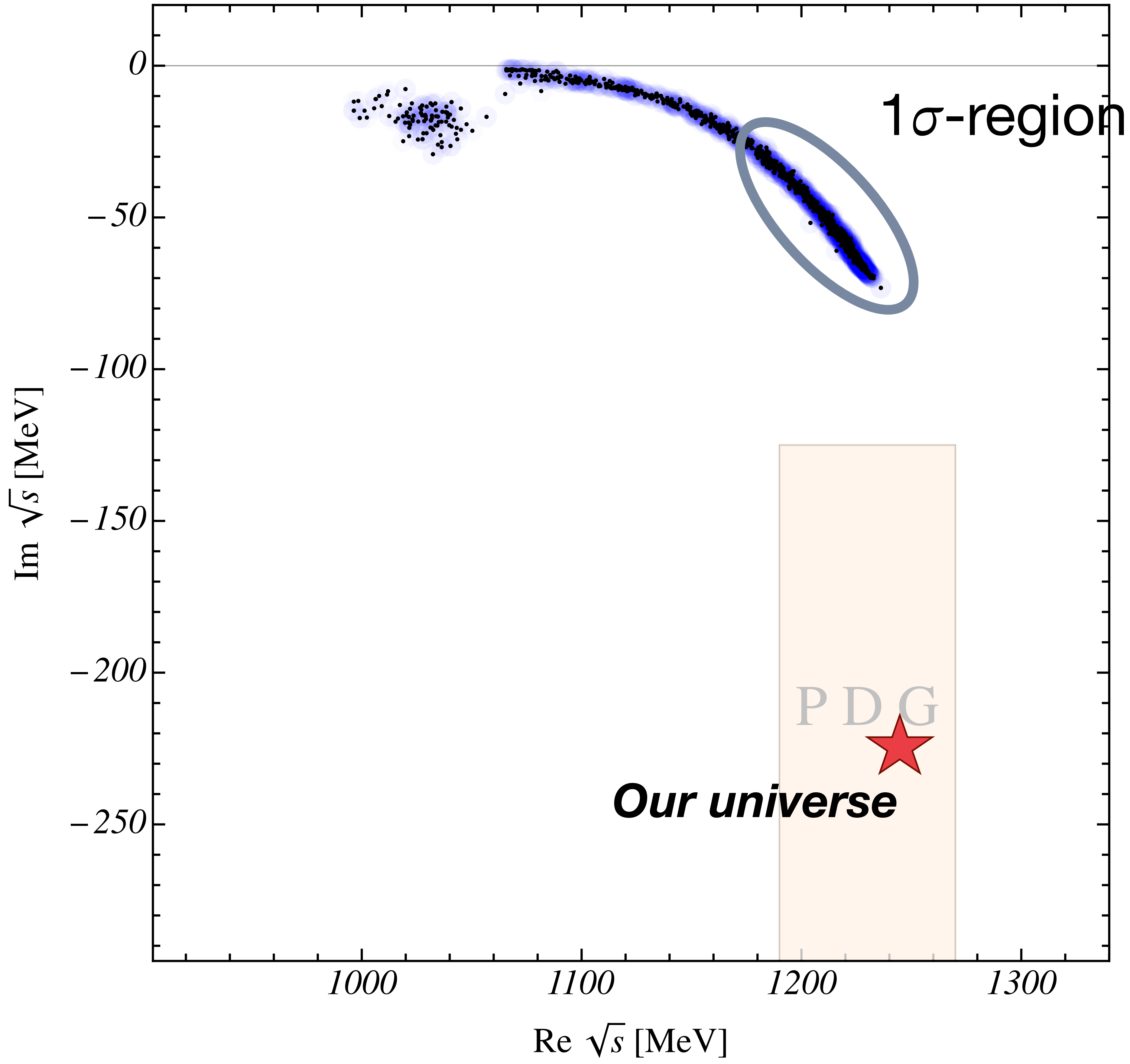
"Heavier Universe"

$$0 = \det\left[2L^3E\left(\tilde{K}_n^{-1} - \Sigma\right) - B - C\right]_{\mathbf{p}'\mathbf{p}}$$

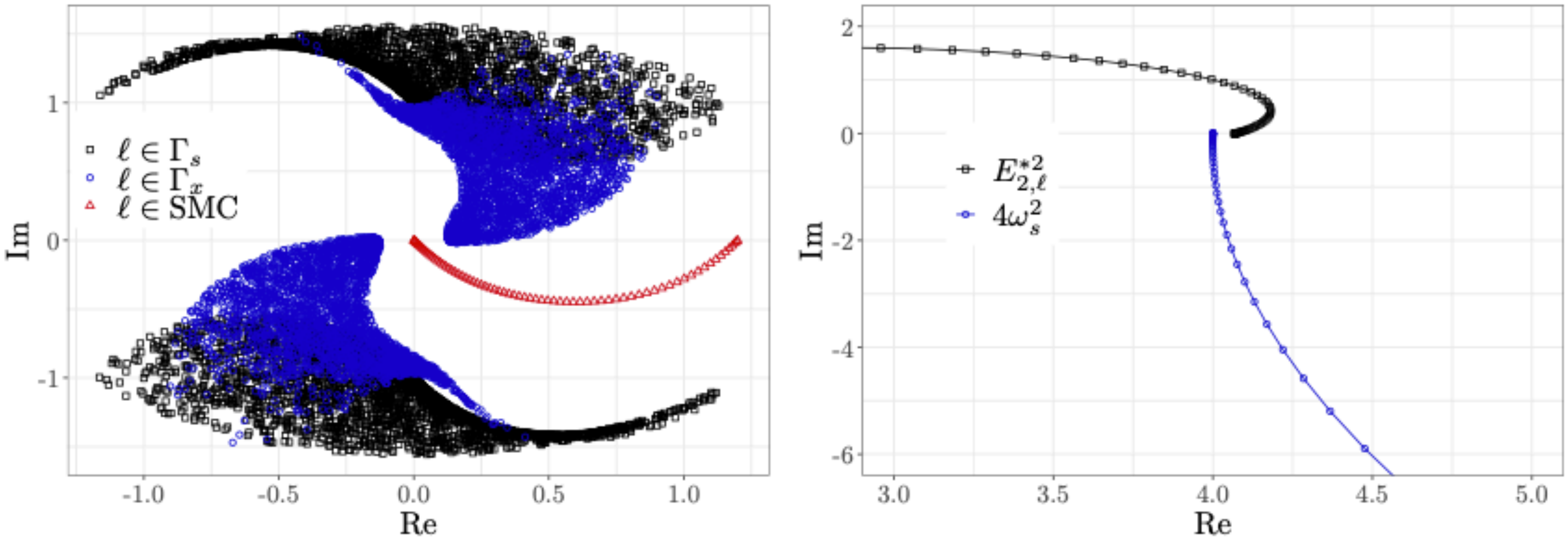
$$T^c = B + C + \int \frac{d^3\ell}{(2\pi)^3}\frac{(B+C)}{2E_l}\frac{1}{\tilde{K}_n^{-1} - \Sigma_n}T^c$$







1) MM/Culver/Döring/Alexandru/Lee/Brett/Sadasivan [GWQCD] Phys.Rev.Lett. 127



Im
Re
-1.0
-0.5
0.0
0.5
1.0
-1
0
1
$\ell \in \Gamma_s$
$\ell \in \Gamma_x$
$\ell \in$ SMC
Im
Re
3.0
3.5
4.0
4.5
5.0
2
0
-2
-4
-6
$E_{2,\ell}^{*2}$
$4\omega_s^2$

Current frontier: 3-body dynamics from LQCD

➥ 3-body Quantization Conditions[1]

➥ RFT / FVU / NREFT

➥ many perturbatively interacting systems are studied[2]

$$0 = \det\left(L^3\left(\tilde{F}/3 - \tilde{F}(\tilde{K}_2^{-1} + \tilde{F} + \tilde{G})^{-1}\tilde{F}\right)^{-1} + K_{\mathrm{df},3}\right)$$ RFT

$$0 = \det\left(B_0 + C_0 - E_L\left(K^{-1}/(32\pi) + \Sigma_L\right)\right)$$ FVU

3-body force

2-body interaction

one-particle exchange

2-body self-energy

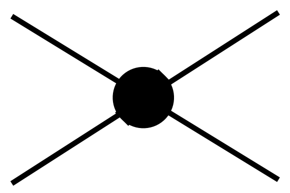
1) Rusetsky, Bedaque, Grießhammer, Sharpe, Meißner, Döring, Hansen, Davoudi, Guo....
Reviews:
Hansen/Sharpe Ann.Rev.Nucl.Part.Sci. 69 (2019);
MM/Döring/Rusetsky Eur.Phys.J.ST 230 (2021);
2) MM/Döring PRL122(2019); Blanton et al. PRL 124 (2020); Hansen et al. PRL 126 (2021); ....

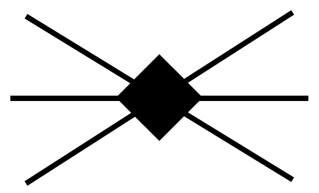
AVOIDED LEVEL CROSS

Variate g(φ₁→φ₀φ₀φ₀) coupling:

- avoided level crossing becomes wider
- RFT and FVU

$$q^* \cot\delta = \frac{1}{aM_0} \qquad C = \frac{c_0}{E_3^3 - m_1^2} + c_1$$





g		a	m_1	c_0	c_1	m'_1	c'_0	c'_1	$\chi^2_{\rm dof}$
5	FVU	−0.1512(9)	3.0229(1)	−0.0188(35)	–	–	–	–	2.9
	RFT	−0.1522(12)	–	–	–	3.0232(2)	31.6(8.4)	–	2.5
	FVU	−0.1569(12)	3.0233(2)	−0.0297(57)	2.29(38)	–	–	–	1.5
	RFT	−0.1571(10)	–	–	–	3.0237(2)	37.6(9.0)	2789(540)	1.5
10	FVU	−0.1521(11)	3.0205(2)	−0.0475(66)	–	–	–	–	1.7
	RFT	−0.1531(13)	–	–	–	3.0212(3)	80(14)	–	1.6
	FVU	−0.1549(16)	3.0205(2)	−0.0595(99)	0.93(41)	–	–	–	1.5
	RFT	−0.1563(27)	–	–	–	3.0213(3)	97(16)	1773(980)	1.4
20	FVU	−0.1444(11)	3.0184(2)	−0.1136(77)	–	–	–	–	1.6
	RFT	−0.1450(17)	–	–	–	3.0199(2)	178(17)	–	1.6
	FVU	−0.1464(14)	3.0183(2)	−0.1363(148)	0.84(39)	–	–	–	1.3
	RFT	−0.1484(16)	–	–	–	3.0200(2)	210(23)	2227(600)	1.2

... same fit quality

... observables determined consistently

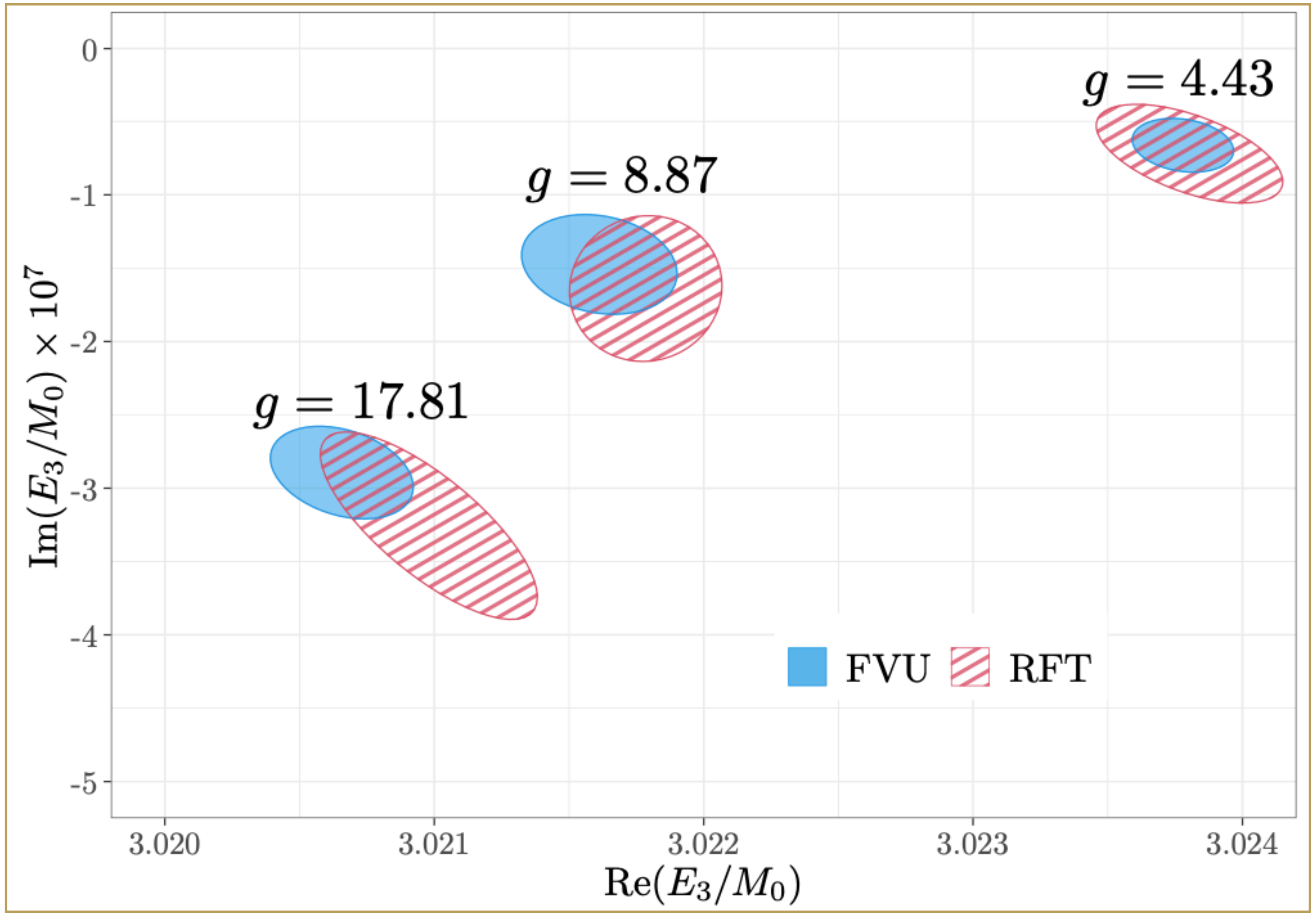
Pole positions

- FVU: complex energy-plane analysis[1]

  -- resonance width grows ~ g^2

  -- avoided level crossing gap >> width

- Similarly from RFT with Breit-Wigner like approximation



1) Sadasivan/MM/.. *Phys.Rev.D* 101 (2020)