3-BODY QUANTIZATION CONDITION IN UNITARY FORMALISM

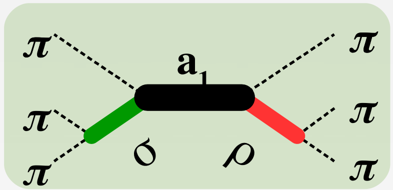
[Eur.Phys.J. A53 (2017) no.9]
[Eur.Phys.J. A53 (2017) no.12]
[Phys.Rev. D97 (2018) no.11]
[arXiv:1807.04746]

Maxim Mai

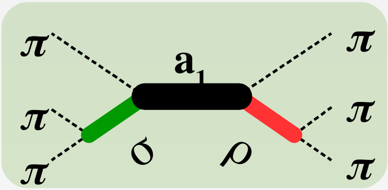
The George Washington University

Deutsche Forschungsgemeinschaft DFG

- Many unsolved questions of QCD involve 3-body channels
  - Roper-puzzle & $\boldsymbol{\pi\pi N}$ channel
  - $\boldsymbol{a_1(1260)} \leftrightarrow \boldsymbol{\pi\rho/\pi\sigma}$ channels $\leftrightarrow$ spectroscopy spin-exotics
  - ***X(3872)*** etc..

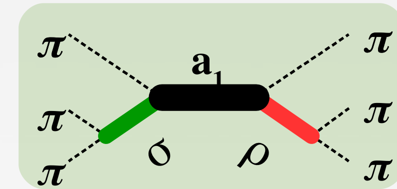


- Many unsolved questions of QCD involve 3-body channels
  - Roper-puzzle & ***ππN*** channel
  - ***a_1(1260)*** ↔ ***πϱ/πσ*** channels ↔ spectroscopy spin-exotics
  - ***X(3872)*** etc..



- Best theoretical tool: **Lattice QCD** → some (preliminary) studies:
  - ***ππN & a_1(1260)*** Lang et al.(2014) Lang et al.(2016)
  - ***πϱ I=2*** [I=2,πρ] Woss et al. (2017)
  - more is under way...

- Many unsolved questions of QCD involve 3-body channels
  - Roper-puzzle & ***ππN*** channel
  - ***a_1(1260)*** ↔ ***πϱ/πσ*** channels ↔ spectroscopy spin-exotics
  - ***X(3872)*** etc..



- Best theoretical tool: **Lattice QCD** → some (preliminary) studies:
  - ***ππN & a_1(1260)*** Lang et al.(2014)Lang et al.(2016)
  - ***πϱ I=2*** [I=2,πρ] Woss et al. (2017)
  - more is under way...

- However,

  Lattice spectrum is discretized → ***mapping to infinite volume*** spectrum

this talk: ***QUANTIZATION CONDITION FOR 3-BODY SYSTEMS***

2-body case

- *one-to-one mapping*
- Various extensions: multi-channels, spin, ...

Lüscher(1986)

Gottlieb,Rummukainen,Feng,Meißner,
Li,Liu,Doring,Briceno,Rusetsky,Bernard…

2-body case

- *one-to-one mapping*
- Various extensions: multi-channels, spin, ...

Lüscher(1986)

Gottlieb,Rummukainen,Feng,Meißner,
Li,Liu,Doring,Briceno,Rusetsky,Bernard…

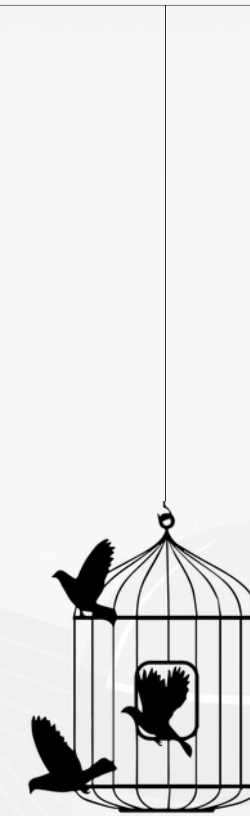
3-body case

- *presumably* no *one-to-one mapping*
  → complex kinematics (8 variables)
  → sub-channel dynamics

- important theoretical developments and *pilot* numerical investigation

Sharpe,Hansen,Briceno,Hammer,Rusetsky,Polejaeva,Griesshammer,Davoudi,Guo…

MM/Doring(2017)
Pang/Hammer/Rusetsky/Wu(2017)
Hansen/Briceno/Sharpe(2018)
Doring/Hammer/MM/Pang/Rusetsky/Wu (2018)



- First data driven study of the volume spectrum
  → ($\pi^+\pi^+$) and ($\pi^+\pi^+\pi^+$) systems
  → comparison with Lattice QCD results

MM/Doring (2018)

> this talk <

UNITARY ISOBAR INF.-VOL. AMPLITUDE

Eur.Phys.J. A53   MM et al. (2017)

1) ***T*** is a sum of a dis/connected parts

UNITARY ISOBAR INF.-VOL. AMPLITUDE

Eur.Phys.J. A53   MM et al. (2017)

1) $\boldsymbol{T}$ is a sum of a dis/connected parts

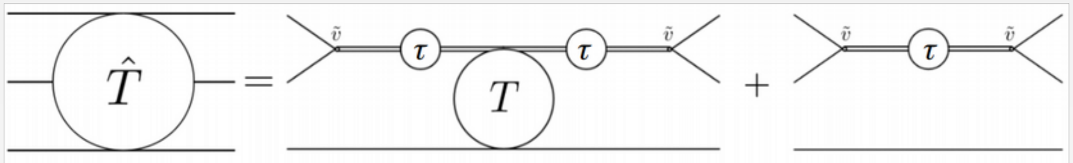
2) **Disconnected part** = spectator + tower of "*isobars*"

- *functions with correct right-hand-singularities for each QN* $\tau(M_{inv})$
- *coupling to asymptotic states: cut-free-function* $v(q,p)$

UNITARY ISOBAR INF.-VOL. AMPLITUDE

Eur.Phys.J. A53 MM et al. (2017)

1) ***T*** is a sum of a dis/connected parts



2) **Disconnected part** = spectator + tower of "*isobars*"

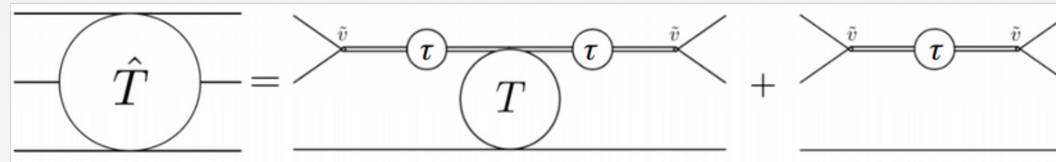
- *functions with correct right-hand-singularities for each QN* ***$\tau(M_{inv})$***
- *coupling to asymptotic states: cut-free-function* ***$v(q,p)$***

3) **Connected part** = general 4d BSE-like equation w.r.t kernel ***$B(p,q;s)$***

UNITARY ISOBAR INF.-VOL. AMPLITUDE

Eur.Phys.J. A53 MM et al. (2017)

1) T is a sum of a dis/connected parts

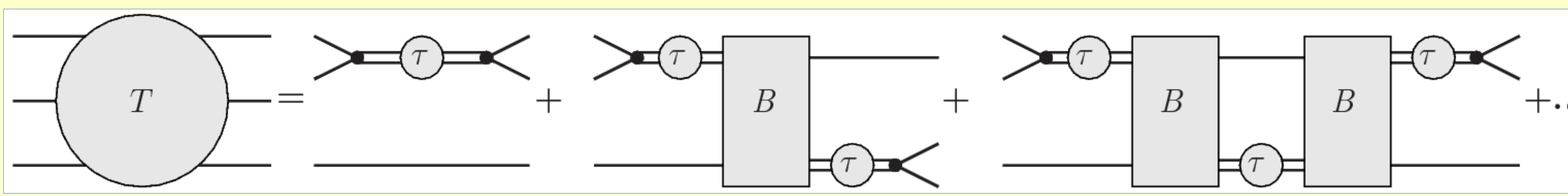


2) **Disconnected part** = spectator + tower of "*isobars*"

- *functions with correct right-hand-singularities for each QN* $\tau(M_{inv})$
- *coupling to asymptotic states: cut-free-function* $v(q,p)$

3) **Connected part** = general 4d BSE-like equation w.r.t kernel $B(p,q;s)$

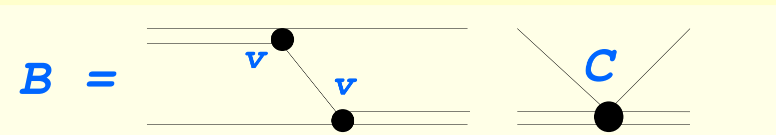
4) **2- and 3-body unitarity** constrains B, τ

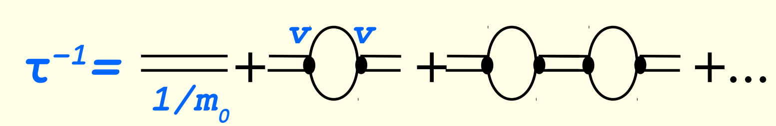


→ relativistic 3d integral-equation

→ useful for phenomenological applications

→ unknowns: v, C, m_0





3-BODY QUANTIZATION CONDITION

Eur.Phys.J. A53 MM/Doring(2017)

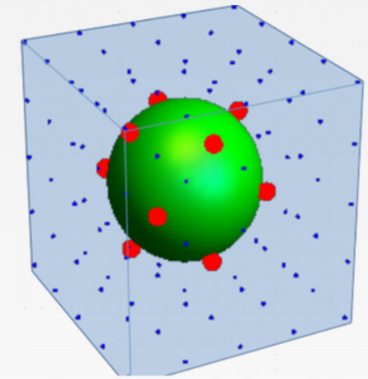
- **Power-law finite-volume effects**

  ↔ on-shell configurations in T ↔ *Im T* ↔ Unitarity is crucial

- Replace integrals by sums:

  $\{E^* | T^{-1}(E^*)=0\}$ = {En.Eigenvalues in a box}

  ⚠ **B is NOT regular → projection to irreps essential**



some useful techniques:

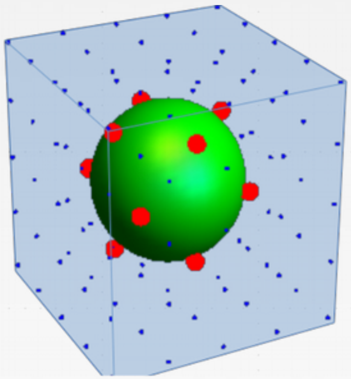
Doring/Hammer/MM/… (2018)

3-BODY QUANTIZATION CONDITION

Eur.Phys.J. A53 MM/Doring(2017)

- **Power-law finite-volume effects**

  ↔ on-shell configurations in T ↔ *Im T* ↔ Unitarity is crucial



- Replace integrals by sums:

  {E|T⁻¹(E*)=0} = {En.Eigenvalues in a box}*

  ⚠ **B is NOT regular → projection to irreps essential**

some useful techniques:
Doring/Hammer/MM/… (2018)

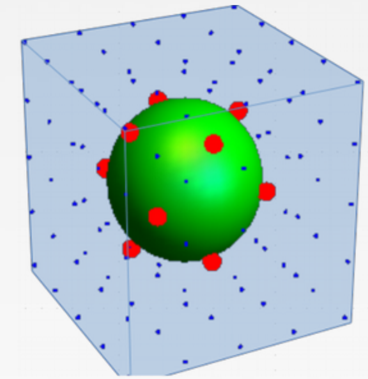
- **Final result** in terms of shells $s^{(\prime)}$ and basis vector index $u^{(\prime)}$

$$\mathrm{Det}\left(\mathbf{B}^{\Gamma\mathbf{ss'}}_{\mathbf{uu'}}(\mathbf{W^2}) + \frac{\mathbf{2E_s\,L^3}}{\vartheta(\mathbf{s})}\tau_{\mathbf{s}}(\mathbf{W^2})^{-1}\delta_{\mathbf{ss'}}\delta_{\mathbf{uu'}}\right) = \mathbf{0}$$

W – total energy
ϑ – multiplicity
L – lattice size
E_s – 1p. energy

3-BODY QUANTIZATION CONDITION

Eur.Phys.J. A53 MM/Doring(2017)



- **Power-law finite-volume effects**

  ↔ on-shell configurations in T ↔ ***Im T*** ↔ Unitarity is crucial

- Replace integrals by sums:

  {E|T⁻¹(E*)=0} = {En.Eigenvalues in a box}*

  ⚠ **B is NOT regular → projection to irreps essential**

some useful techniques:

Doring/Hammer/MM/… (2018)

➢ **Final result** in terms of shells $s^{(l)}$ and basis vector index $u^{(l)}$

$$\mathrm{Det}\left(\mathbf{B}^{\Gamma \mathbf{ss}'}_{\mathbf{uu}'}(\mathbf{W^2})+\frac{\mathbf{2E_s\,L^3}}{\vartheta(\mathbf{s})}\tau_{\mathbf{s}}(\mathbf{W^2})^{-1}\delta_{\mathbf{ss}'}\delta_{\mathbf{uu}'}\right)=\mathbf{0}$$

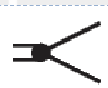
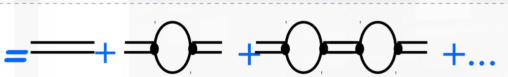
W – total energy

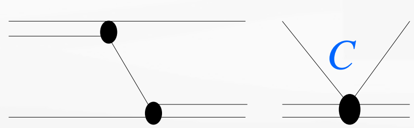
ϑ – multiplicity

L – lattice size

E_s – 1p. energy

- Possible work-flow:

  1) Fix $v=$  & $\tau^{-1}=$  to 2-body channel (Lattice or Exp. data)

  2) Fix C in $B=$  to 3-body data (Lattice or Exp. data)

PHYSICAL APPLICATION

arXiv:1807.04746 MM/Doring(2018)

- Interesting system: $\pi^+\pi^+\pi^+$
  - LatticeQCD results for ground level available for $\pi^+\pi^+$ & $\pi^+\pi^+\pi^+$ — Detmold et al.(2008)
  - Repulsive channel → ***Q:*** *does the "isobar" picture hold?*
  - ***L=2.5 fm, m_π=291/352/491/591 MeV*** → ***BonusQ:*** *chiral extrapolation in 3body system?*

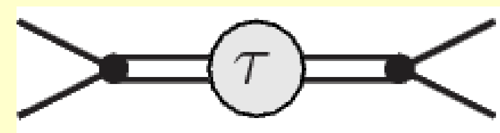
PHYSICAL APPLICATION

arXiv:1807.04746 MM/Doring(2018)

- Interesting system: $\pi^+\pi^+\pi^+$
  - LatticeQCD results for ground level available for $\pi^+\pi^+$ & $\pi^+\pi^+\pi^+$ — Detmold et al.(2008)
  - Repulsive channel → ***Q:*** *does the "isobar" picture hold?*
  - ***L=2.5 fm, m_π=291/352/491/591 MeV*** → ***BonusQ:*** *chiral extrapolation in 3body system?*

I. 2-body subchannel:

- one-channel problem: $\pi\pi$-system in S-wave, I=2
- 2-body amplitude consistent with 3-body one



$$\frac{-\lambda^2/(32\pi)}{\sigma - M_0^2 - \sum_\pm \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{\lambda^2}{4E_{\boldsymbol{k}}\sqrt{\sigma}(\sqrt{\sigma}\pm 2E_{\boldsymbol{k}})}}$$

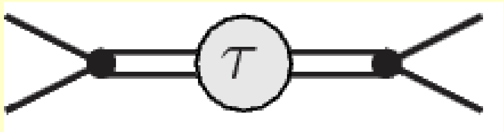
PHYSICAL APPLICATION

arXiv:1807.04746 MM/Doring(2018)

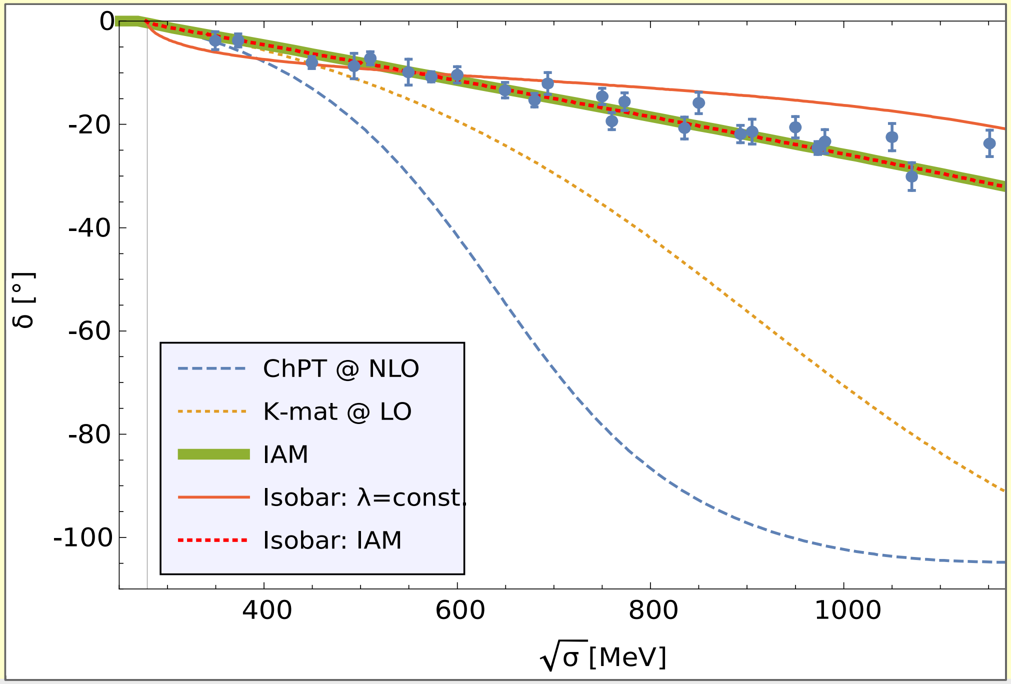
- Interesting system: $\pi^+\pi^+\pi^+$
  - LatticeQCD results for ground level available for $\pi^+\pi^+$ & $\pi^+\pi^+\pi^+$ Detmold et al.(2008)
  - Repulsive channel → ***Q:*** *does the "isobar" picture hold?*
  - ***L=2.5 fm, m_π=291/352/491/591 MeV*** → ***BonusQ:*** *chiral extrapolation in 3body system?*

I. 2-body subchannel:

- one-channel problem: $\pi\pi$-system in S-wave, I=2
- 2-body amplitude consistent with 3-body one



$$\frac{-\lambda^2/(32\pi)}{\sigma - M_0^2 - \sum_\pm \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{\lambda^2}{4E_{\boldsymbol{k}}\sqrt{\sigma}(\sqrt{\sigma}\pm 2E_{\boldsymbol{k}})}}$$



1) Fix λ, M_0 to exp. data

☹ incoorrect m_π behavior!

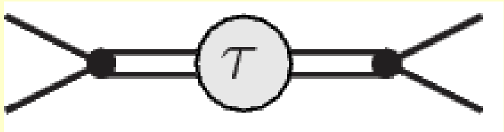
PHYSICAL APPLICATION

arXiv:1807.04746 MM/Doring(2018)

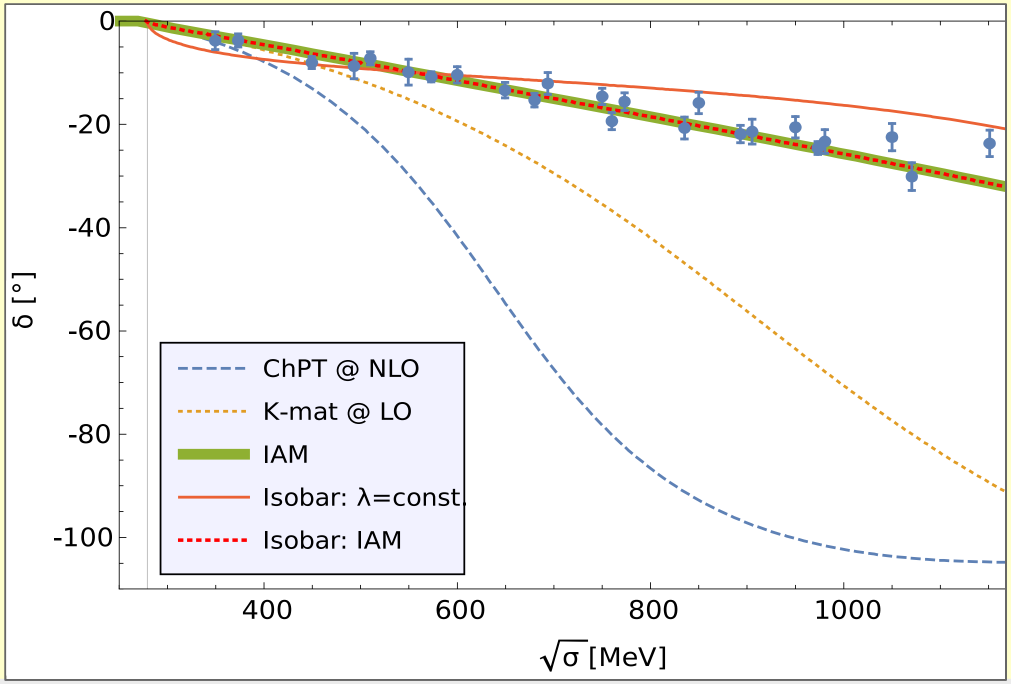
- Interesting system: $\pi^+\pi^+\pi^+$
  - LatticeQCD results for ground level available for $\pi^+\pi^+$ & $\pi^+\pi^+\pi^+$ Detmold et al.(2008)
  - Repulsive channel → *Q: does the "isobar" picture hold?*
  - ***L=2.5 fm, m_π=291/352/491/591 MeV*** → ***BonusQ:*** *chiral extrapolation in 3body system?*

I. 2-body subchannel:

- one-channel problem: $\pi\pi$-system in S-wave, I=2
- 2-body amplitude consistent with 3-body one



$$\frac{-\lambda^2/(32\pi)}{\sigma - M_0^2 - \sum_\pm \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{\lambda^2}{4E_{\boldsymbol{k}}\sqrt{\sigma}(\sqrt{\sigma}\pm 2E_{\boldsymbol{k}})}}$$



1) ~~**Fix λ, M_0 to exp. data**~~

☹ incoorrect m_π behavior!

2) **Chiral NLO & K-matrix** Gasser/Leutwyler(1984)

☹ works badly for high energies

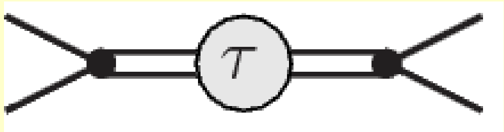
PHYSICAL APPLICATION

arXiv:1807.04746 MM/Doring(2018)

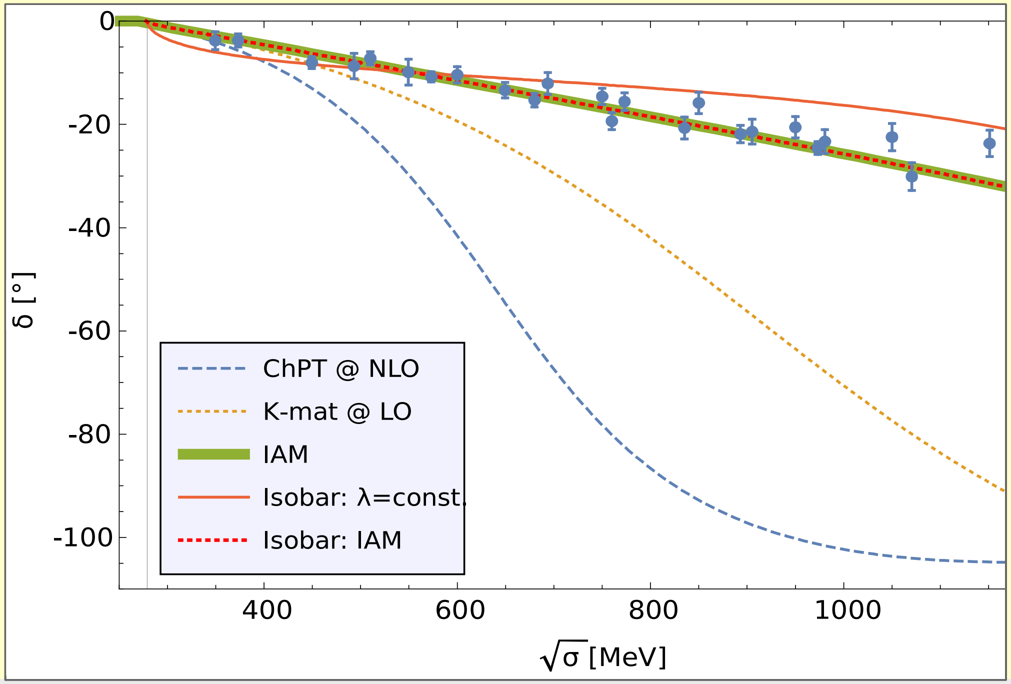
- Interesting system: $\pi^+\pi^+\pi^+$
  - LatticeQCD results for ground level available for $\pi^+\pi^+$ & $\pi^+\pi^+\pi^+$ Detmold et al.(2008)
  - Repulsive channel → *Q: does the "isobar" picture hold?*
  - ***L=2.5 fm, m_π=291/352/491/591 MeV*** → ***BonusQ:*** *chiral extrapolation in 3body system?*

I. 2-body subchannel:

- one-channel problem: $\pi\pi$-system in S-wave, I=2
- 2-body amplitude consistent with 3-body one



$$\frac{T_{\rm LO}^2}{T_{\rm LO}-T_{\rm NLO}}$$



1) ~~Fix λ, M_0 to exp. data~~

☹ incoorrect m_π behavior!

2) ~~Chiral NLO & K-matrix~~

☹ works badly for high energies

3) **Inverse Amplitude** Truong(1988)

☺ correct σ & m_π behavior

☺ parameters known

Gasser/Leutwyler(1984)

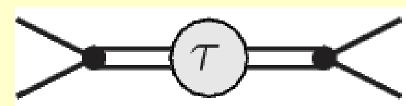
PHYSICAL APPLICATION

arXiv:1807.04746 MM/Doring(2018)

- Interesting system: $\pi^+\pi^+\pi^+$
  - LatticeQCD results for ground level available for $\pi^+\pi^+$ & $\pi^+\pi^+\pi^+$ Detmold et al.(2008)
  - Repulsive channel → ***Q:*** *does the "isobar" picture hold?*
  - ***L=2.5 fm, m_π=291/352/491/591 MeV*** → ***BonusQ:*** *chiral extrapolation in 3body system?*

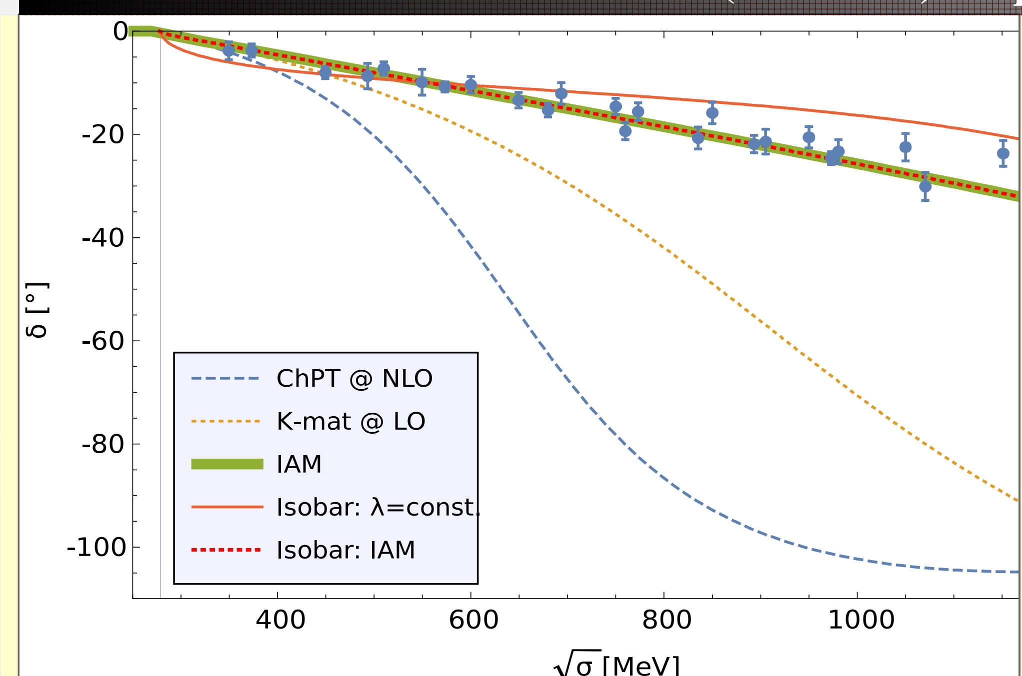
I. 2-body subchannel:

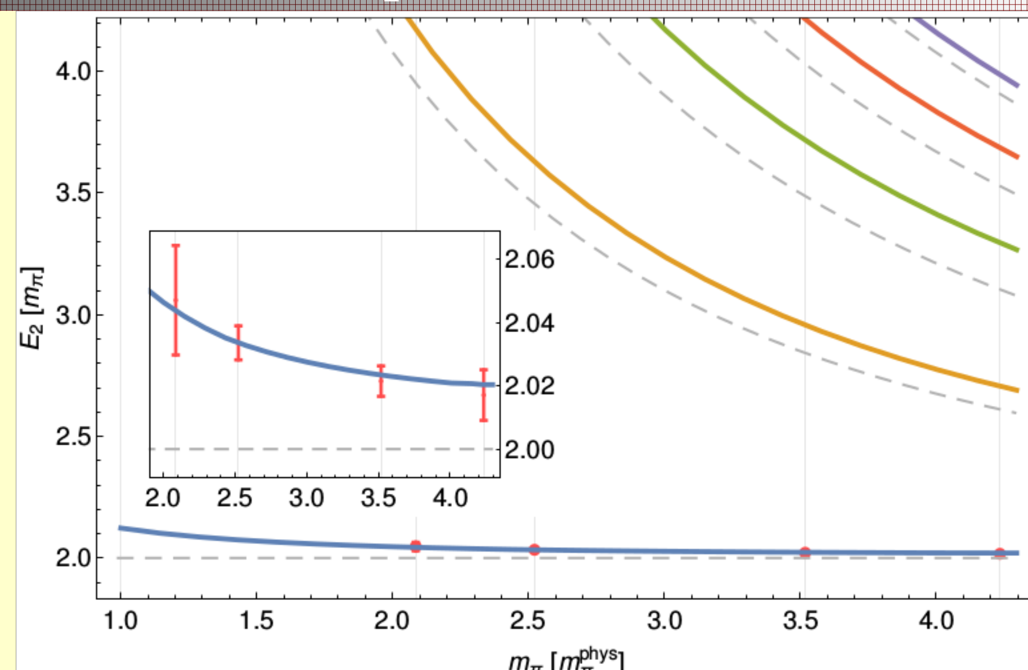
- one-channel problem: $\pi\pi$-system in S-wave, I=2
- 2-body amplitude consistent with 3-body one



$$\frac{T_{\mathrm{LO}}^2}{T_{\mathrm{LO}} - T_{\mathrm{NLO}}}$$

discretize (Lüscher) → predicted fin-vol. spectrum





PHYSICAL APPLICATION

arXiv:1807.04746 MM/Doring(2018)

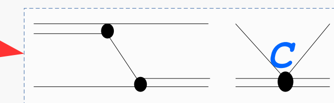
- Interesting system: $\pi^+\pi^+\pi^+$
  - LatticeQCD results for ground level available for $\pi^+\pi^+$ & $\pi^+\pi^+\pi^+$ Detmold et al.(2008)
  - Repulsive channel → *Q: does the "isobar" picture hold?*
  - ***L=2.5 fm, m_π=291/352/491/591 MeV*** → ***BonusQ:*** *chiral extrapolation in 3body system?*

II. 3-body spectrum

Remaining unknown: C

QUANTIZATION CONDITION

$$\mathrm{Det}\left(\mathbf{B}^{\Gamma\mathbf{ss}'}_{\mathbf{uu}'}(\mathbf{W^2})+\frac{\mathbf{2E_s\,L^3}}{\vartheta(\mathbf{s})}\tau_\mathbf{s}(\mathbf{W^2})^{-1}\delta_{\mathbf{ss}'}\delta_{\mathbf{uu}'}\right)=\mathbf{0}$$



- *genuine (momenta-dependent) 3-body "force"*
- *simplest case:* $\boxed{C_{\boldsymbol{qp}} = c\,\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})}$

PHYSICAL APPLICATION

arXiv:1807.04746 MM/Doring(2018)

- Interesting system: $\pi^+\pi^+\pi^+$
  - LatticeQCD results for ground level available for $\pi^+\pi^+$ & $\pi^+\pi^+\pi^+$ Detmold et al.(2008)
  - Repulsive channel → ***Q:*** *does the "isobar" picture hold?*
  - ***L=2.5 fm, m_π=291/352/491/591 MeV*** → ***BonusQ:*** *chiral extrapolation in 3body system?*

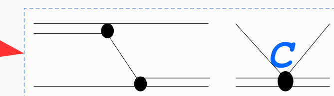
II. 3-body spectrum

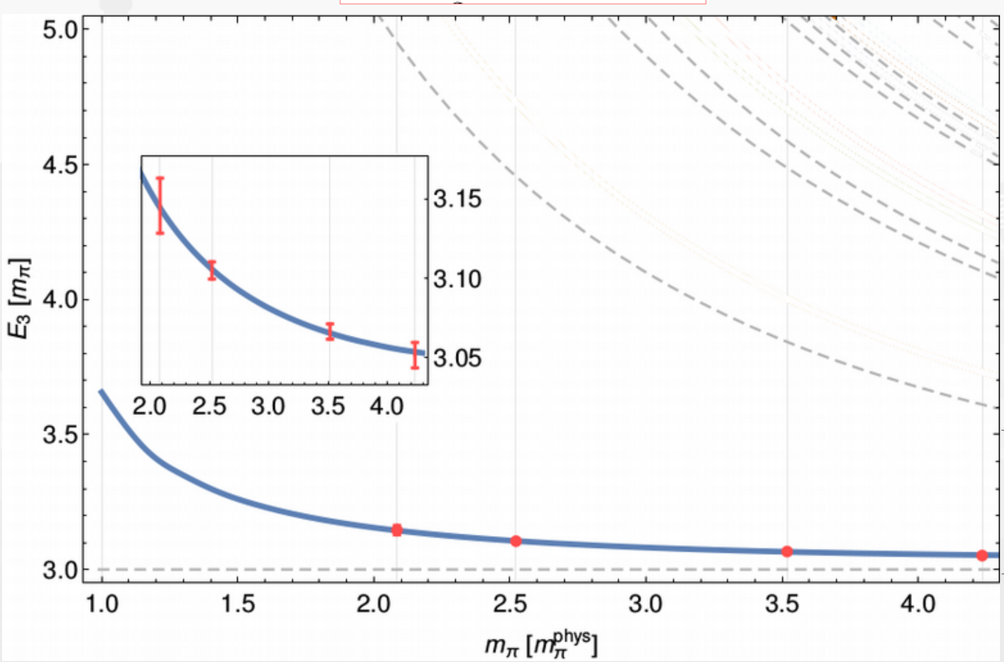
Remaining unknown: ***C***

- *genuine (momenta-dependent) 3-body "force"*
- *simplest case:* $\boxed{C_{\boldsymbol{qp}} = c\,\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})}$

QUANTIZATION CONDITION

$$\mathrm{Det}\left(\mathbf{B}^{\Gamma \mathbf{ss}'}_{\mathbf{uu}'}(\mathbf{W^2}) + \frac{\mathbf{2E_s\,L^3}}{\vartheta(\mathbf{s})}\tau_{\mathbf{s}}(\mathbf{W^2})^{-1}\delta_{\mathbf{ss}'}\delta_{\mathbf{uu}'}\right) = \mathbf{0}$$





Fit *C* to NPLQCD ground state level
→ ***C=0.2 ± 1.5·10^-10***

PHYSICAL APPLICATION

arXiv:1807.04746 MM/Doring(2018)

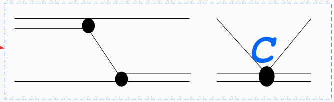
- Interesting system: $\pi^+\pi^+\pi^+$
  - LatticeQCD results for ground level available for $\pi^+\pi^+$ & $\pi^+\pi^+\pi^+$ Detmold et al.(2008)
  - Repulsive channel → *Q: does the "isobar" picture hold?*
  - ***L=2.5 fm, m_π=291/352/491/591 MeV*** → ***BonusQ:*** *chiral extrapolation in 3body system?*

II. 3-body spectrum

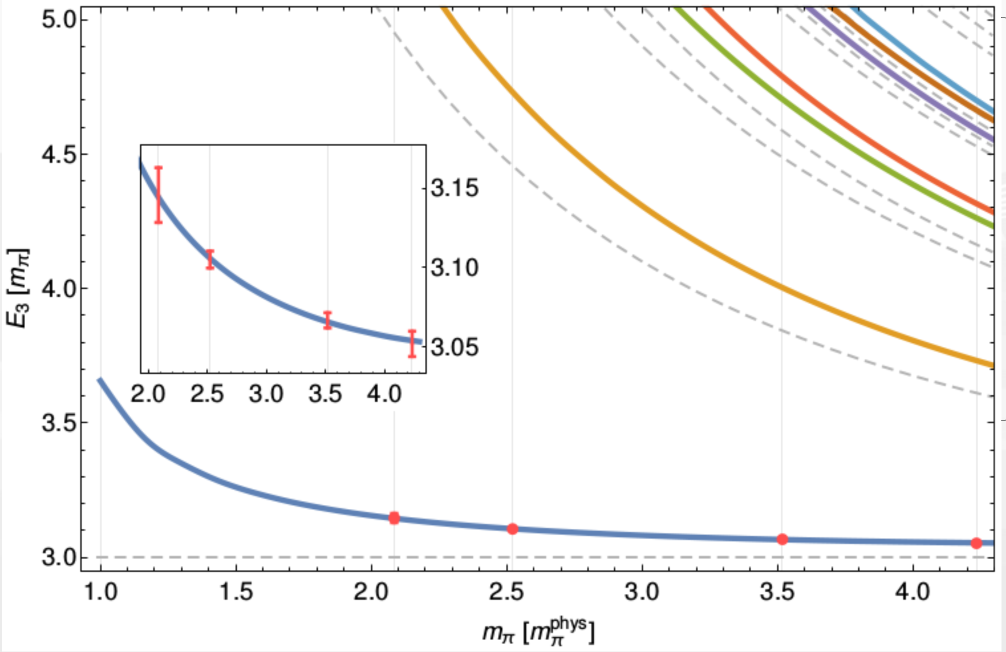
QUANTIZATION CONDITION

$$\mathrm{Det}\left(\mathbf{B}^{\Gamma ss'}_{\mathbf{uu'}}(\mathbf{W^2}) + \frac{\mathbf{2E_s\,L^3}}{\vartheta(\mathbf{s})}\tau_{\mathbf{s}}(\mathbf{W^2})^{-1}\delta_{\mathbf{ss'}}\delta_{\mathbf{uu'}}\right) = \mathbf{0}$$

Remaining unknown: C



- *genuine (momenta-dependent) 3-body "force"*
- *simplest case:* $\boxed{C_{\boldsymbol{qp}} = c\,\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})}$



Predict exited spectrum:

→ novel pattern

1/1 of interacting/non-interacting lvls

→ all QC-poles are simple

→ chiral extrapolation to phys point

"LIBERATION/CAPTURING OF 3 BIRDS (PARTICLES)"= SUMMARY

"Three-body Unitarity with Isobars Revisited"

[Eur.Phys.J. A53 (2017) no.9]

- *Parametrization via 2-body sub-channel amplitudes ("isobars")*
- *Relativistic integral equation*
- *Phenomenological applications in progress…*

"Three-body Unitarity in the Finite Volume"

[Eur.Phys.J. A53 (2017) no.9, 177]
[Phys.Rev. D97 (2018) no.11]

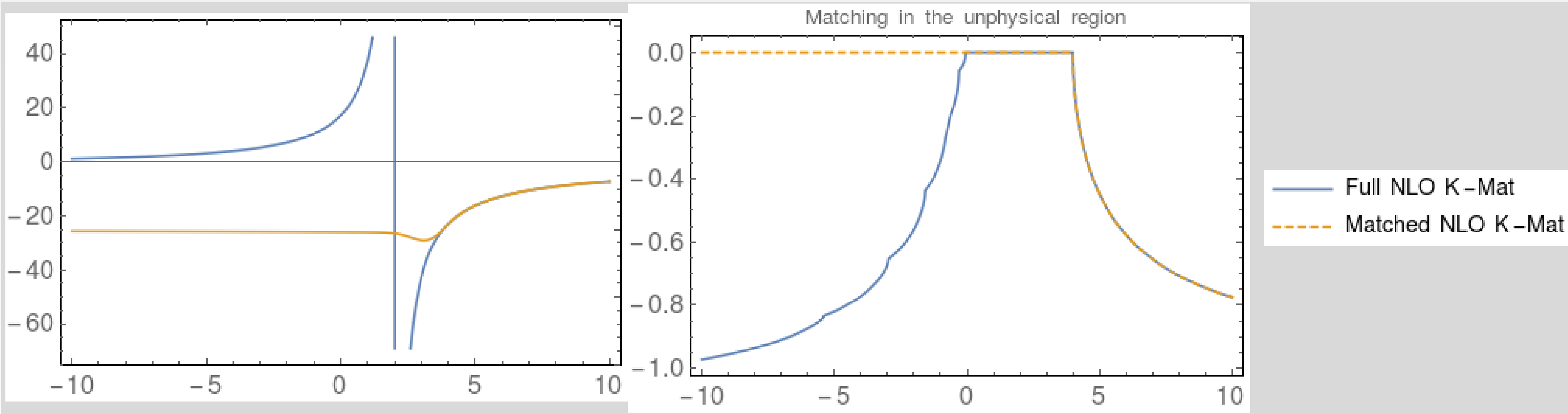
- *Discretization & Projection to irreps of $\boldsymbol{O_h}$ leads to 3body QC*
- *Numerical toy-examples explored*
- *Extension to multi-channels in progress…*

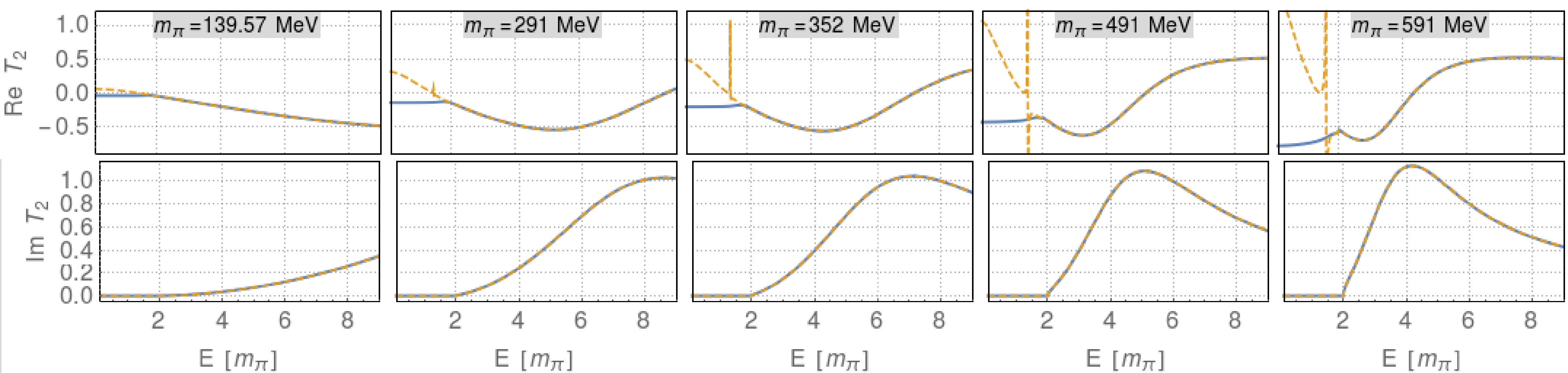
"Finite-volume spectrum of $\pi^+\pi^+$ and $\pi^+\pi^+\pi^+$ systems"

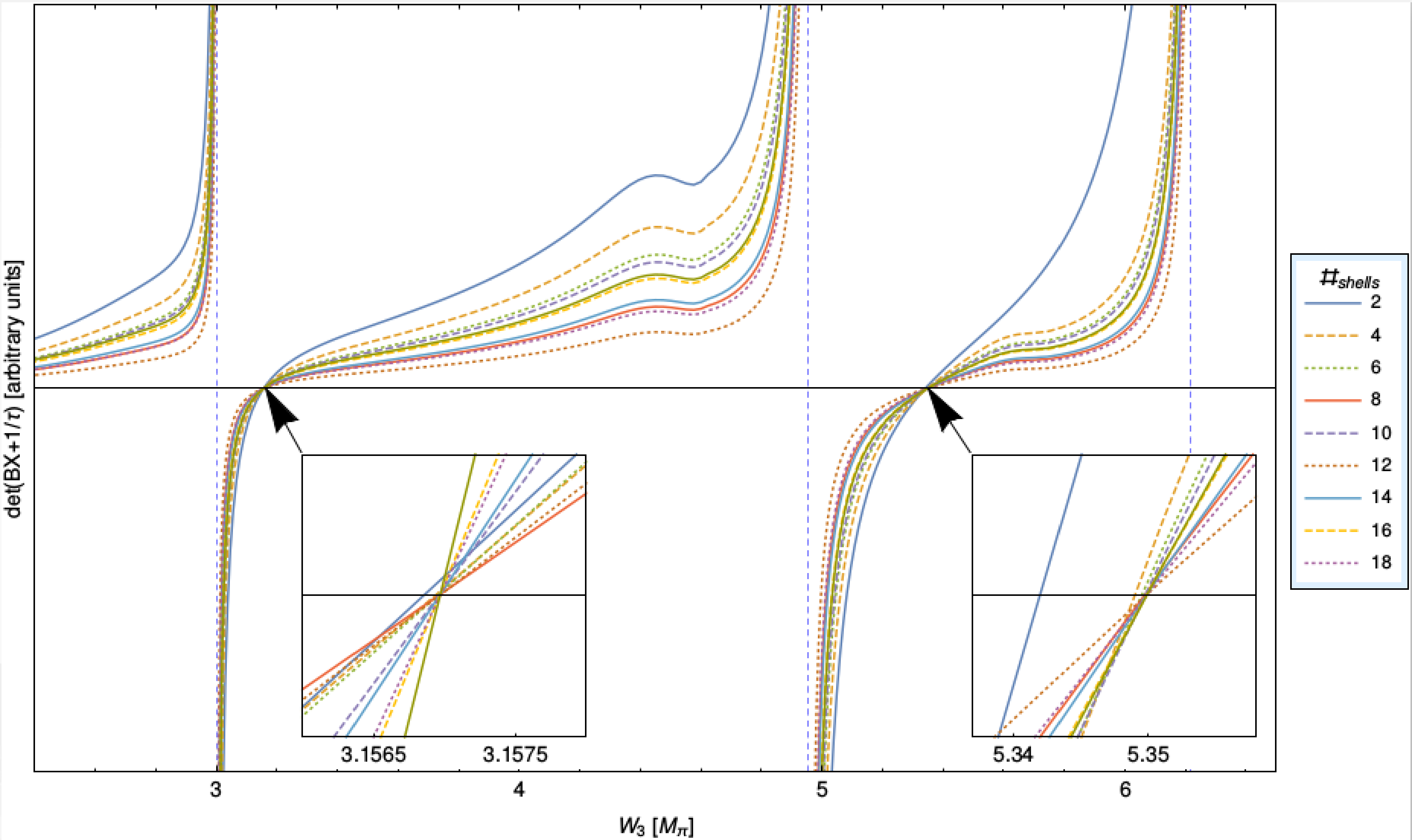
[arXiv:1807.04746]

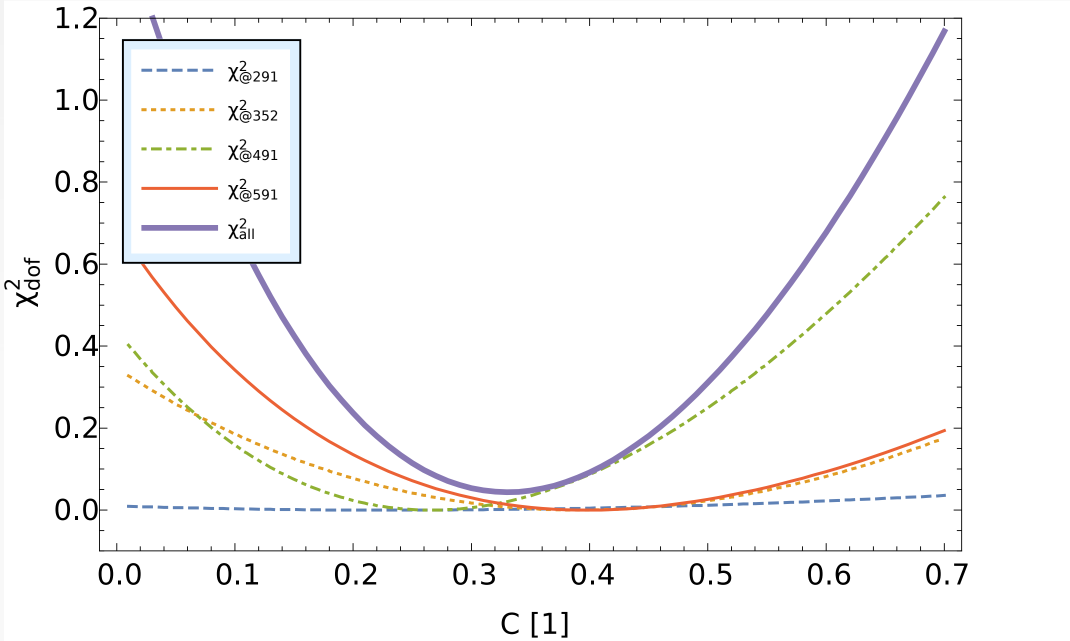
- *(excited spectrum) of $\pi^+\pi^+$ & $\pi^+\pi^+\pi^+$ systems* ***predicted*** *from 3b QC*
- *ground level compared with NPLQCD results*
- *3-body fin-vol. spectrum features explored*
- *Predictions at phys. pion mass*
  - *Outlook: N*(1440), …*

BACKUP


Matching in the unphysical region
Full NLO K-Mat
Matched NLO K-Mat


m_π =139.57 MeV
m_π =291 MeV
m_π =352 MeV
m_π =491 MeV
m_π =591 MeV
Re T_2
Im T_2
E [m_π]


det(BX+1/τ) [arbitrary units]
W_3 [M_π]
3
4
5
6
3.1565
3.1575
5.34
5.35
#shells
2
4
6
8
10
12
14
16
18


$\chi^2_{@291}$
$\chi^2_{@352}$
$\chi^2_{@491}$
$\chi^2_{@591}$
χ^2_{all}
χ^2_{dof}
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0.0 0.1 0.2 0.3 0.4 0.5 0.6 0.7
C [1]

m_π [MeV]	139.57	291	352	491	591
E_2^1 $[m_\pi]$	$2.1228^{+0.0068}_{-0.0069}$	$2.0437^{+0.0071}_{-0.0086}$	$2.0334^{+0.0076}_{-0.0086}$	$2.0233^{+0.0105}_{-0.0098}$	$2.0204^{+0.0200}_{-0.0106}$
Refs. [24, 25]	–	$\mathbf{2.0471(27)(65)}$	$\mathbf{2.0336(22)(22)}$	$\mathbf{2.0215(16)(13)}$	$\mathbf{2.0171(16)(19)}$
E_2^2 $[m_\pi]$	–	–	$3.6245^{+0.0746}_{-0.0299}$	$2.9556^{+0.0728}_{-0.0263}$	$2.7045^{+0.0827}_{-0.0271}$
E_2^3 $[m_\pi]$	–	–	–	$3.7114^{+0.1482}_{-0.0737}$	$3.2911^{+0.1241}_{-0.0688}$
E_2^4 $[m_\pi]$	–	–	–	–	$3.6802^{+0.0707}_{-0.0902}$
E_2^5 $[m_\pi]$	–	–	–	–	$3.9829^{+0.0500}_{-0.0299}$
E_3^1 $[m_\pi]$	$3.6564^{+0.1014}_{-0.0847}$	$^*3.1444^{+0.0171}_{-0.0192}$	$^*3.1058^{+0.0091}_{-0.0147}$	$^*3.0655^{+0.0029}_{-0.0095}$	$^*3.0537^{+0.0048}_{-0.0119}$
Refs. [24, 25]	–	$\mathbf{3.1458(49)(125)}$	$\mathbf{3.1050(27)(27)}$	$\mathbf{3.0665(26)(22)}$	$\mathbf{3.0516(27)(53)}$
E_3^2 $[m_\pi]$	–	–	$4.7301^{+0.1577}_{-0.1027}$	$4.0031^{+0.0196}_{-0.1836}$	$3.7315^{+0.0309}_{-0.0742}$
E_3^3 $[m_\pi]$	–	–	–	$4.7043^{+0.0126}_{-0.5923}$	$4.2621^{+0.0001}_{-0.1739}$
E_3^4 $[m_\pi]$	–	–	–	$4.7890^{+0.0506}_{-0.1722}$	$4.3155^{+0.0837}_{-0.1341}$
E_3^5 $[m_\pi]$	–	–	–	–	$4.5913^{+0.0001}_{-0.1995}$
E_3^6 $[m_\pi]$	–	–	–	–	$4.6634^{+0.0001}_{-0.1070}$
E_3^7 $[m_\pi]$	–	–	–	–	$4.6995^{+0.0001}_{-0.0661}$